

INJECTIVITY THEOREM FOR PSEUDO-EFFECTIVE LINE BUNDLES AND ITS APPLICATIONS

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ABSTRACT. We formulate and establish a generalization of Kollár’s injectivity theorem for adjoint bundles twisted by a suitable multiplier ideal sheaf. As applications, we generalize Kollár’s vanishing theorem, Kollár’s torsion-freeness, generic vanishing theorem, and so on, for pseudo-effective line bundles. Our approach is not Hodge theoretic but is analytic since we treat singular hermitian metrics with arbitrary singularities. For the proof of the main injectivity theorem, we use the theory of harmonic integrals on noncompact Kähler manifolds. For applications, we prove a Bertini-type theorem on the restriction of multiplier ideal sheaves to general members of free linear systems, which seems to be of independent interest.

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1. INTRODUCTION

The Kodaira vanishing theorem [Kod] is one of the most celebrated results in complex geometry, and it has many important and useful generalizations, for example, the Kawamata–Viehweg vanishing theorem, the Nadel vanishing theorem, Kollár’s injectivity theorem, and so on (see, for example, [F9, Chapter 3]). Kodaira’s original proof is based on his theory of harmonic integrals on compact complex manifolds and the proof of the Nadel vanishing theorem is based on the L^2 -method for the $\bar{\partial}$ -equation. Now we know that we can quickly prove the Kodaira vanishing theorem for smooth complex projective varieties by using the Hodge theory (see [L1, Section 4.2]). In [Kol1], Kollár obtained his famous injectivity theorem, which is one of the most important generalizations of the Kodaira vanishing theorem for smooth complex projective varieties. His proof in [Kol1] is Hodge theoretic. After Kollár’s important work, Enoki [En] recovered and generalized

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Kollár's injectivity theorem as an easy application of the theory of harmonic integrals on compact Kähler manifolds. As we mentioned above, we have two important approaches to consider various generalizations of the Kodaira vanishing theorem: One approach is the Hodge theory and the other is the transcendental method based on the theory of harmonic integrals or the L^2 -method for the $\bar{\partial}$ -equation. To the best knowledge of the authors, there are no analytic proofs of the following important and useful injectivity theorem, which easily follows from the Hodge theory.

Theorem 1.1 (see [EV] and [Kol2]). *Let L be a holomorphic line bundle on a smooth projective variety X . Assume that there is $\Delta \in |L^{\otimes m}|$ for some positive integer $m \geq 2$ such that $\text{Supp } \Delta$ is a simple normal crossing divisor on X and that the coefficients of Δ are less than m . Let D be an effective Cartier divisor on X with $\text{Supp } D \subset \text{Supp } \Delta$. Then the map*

$$H^i(X, \omega_X \otimes L) \rightarrow H^i(X, \omega_X \otimes L \otimes \mathcal{O}_X(D))$$

induced by the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$ is injective for every i , where ω_X is the canonical bundle of X .

We can easily recover Kollár's original injectivity theorem in [Kol1] from Theorem 1.1. The authors think that the precise relationship between the Hodge theoretic approach and the transcendental method is not clear yet and is still mysterious. From the Hodge theoretic viewpoint, we have already obtained a satisfactory generalization of Theorem 1.1, whose proof heavily depends on the theory of mixed Hodge structures on cohomology with compact support (see, for example, [F9, Chapter 5]). It has many applications suitable for the minimal model program (see [F1], [F2], [F3], [F6], [F7], [F8], [F9], [F10], [F12], [F13], [F14], and so on). There is also an approach from Saito's theory of Hodge modules (see [Wu]). On the other hand, also from the analytic viewpoint, we already have some generalizations of Kollár's original injectivity theorem (see, for example, [En], [Ta], [O2], [F4], [F5], [MaS1], [MaS2], and [MaS4]). However, there seems to be room for further research from the analytic viewpoint. The transcendental method often provides some very powerful tools not only in complex geometry but also in algebraic geometry. Therefore it is natural and of interest to study various vanishing theorems and related topics by using the transcendental method.

In this paper, we pursue the transcendental approach further and establish a generalization of Kollár's injectivity theorem for adjoint bundles twisted by a suitable multiplier ideal sheaf (see Theorem A below). Moreover, we discuss some related topics and give various applications. Since we adopt the transcendental method, we can formulate all the results in this paper for singular hermitian metrics and (quasi-)plurisubharmonic functions with *arbitrary* singularities. This is one of the main advantages of our approach in this paper. Interestingly, we sometimes have to deal with singular hermitian metrics with nonalgebraic singularities for several important applications in birational geometry even when we consider problems in algebraic geometry (see, for example, [Si], [Pa], [DHP], [GM], and [LP]). Therefore it is worth formulating and proving various results for singular hermitian metrics with arbitrary singularities although they are much more complicated than singular hermitian metrics with only algebraic singularities. We recommend the reader to see the survey articles [F11], [MaS3], and [MaS7] for our recent results and some related problems. We note that there are many related results which are not mentioned here by lack of space and by our ignorance. We apologize to all those whose works are not adequately referred in this paper.

Before we explain the main results of this paper, we recall the definition of pseudo-effective line bundles on compact complex manifolds.

Definition 1.2 (Pseudo-effective line bundles). Let F be a holomorphic line bundle on a compact complex manifold X . We say that F is pseudo-effective if there exists a singular hermitian metric h on F with $\sqrt{-1}\Theta_h(F) \geq 0$. When X is projective, it is well known that F is pseudo-effective if and only if F is pseudo-effective in the usual sense, that is, $F^{\otimes m} \otimes H$ is big for any ample line bundle H on X and any positive integer m .

Roughly speaking, in this paper, we will prove Kollár's injectivity, vanishing, and torsion-free theorems and generic vanishing theorem for $\omega_X \otimes F \otimes \mathcal{J}(h)$ instead of ω_X , where ω_X is the canonical bundle of X , F is a pseudo-effective line bundle on X , and $\mathcal{J}(h)$ is the multiplier ideal sheaf associated to a singular hermitian metric h . Moreover, our arguments work for $\omega_X \otimes E \otimes F \otimes \mathcal{J}(h)$, where E is any Nakano semipositive vector bundle on X , with only some minor modifications (see Theorem 1.11).

1.1. Main results. Let us explain the main results of this paper (Theorems A, B, C, D, E, and F). Theorem A is the most important result. We will see that the other results follow from Theorem A (see Proposition 1.9).

Theorem A (Enoki-type injectivity). *Let F be a holomorphic line bundle on a compact Kähler manifold X and let h be a singular hermitian metric on F . Let M be a holomorphic line bundle on X and let h_M be a smooth hermitian metric on M . Assume that*

$$\sqrt{-1}\Theta_{h_M}(M) \geq 0 \quad \text{and} \quad \sqrt{-1}(\Theta_h(F) - t\Theta_{h_M}(M)) \geq 0$$

for some $t > 0$. Let s be a nonzero global section of M . Then the map

$$\times s : H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h)) \rightarrow H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes M)$$

induced by $\otimes s$ is injective for every i , where ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h .

Remark 1.3. Let L be a semipositive line bundle on X , that is, it admits a smooth hermitian metric with semipositive curvature. If $F \simeq L^{\otimes m}$ and $M \simeq L^{\otimes k}$ for some positive integers m and k , then Theorem A recovers the original Enoki injectivity theorem (see [En, Theorem 0.2]).

The proof of Theorem A is an improvement of the arguments in [MaS4] and is based on a combination of the theory of harmonic integrals and the L^2 -method for the $\bar{\partial}$ -equation. Theorem A looks out of reach of the Hodge theory because we assume nothing on the singularities of h . If h is smooth on a nonempty Zariski open set, then Theorem A follows from [F4, Theorem 1.2], which is also analytic. Theorem A can be seen as a generalization not only of Enoki's injectivity theorem but also of the Nadel vanishing theorem. Indeed, it leads to Demailly's original formulation of the Nadel vanishing theorem (see Theorem 1.4 below). In Section 4, we will explain how to reduce Theorem 1.4 to Theorem A for the reader's convenience.

Theorem 1.4 (Nadel vanishing theorem due to Demailly: [D2, Theorem 4.5]). *Let V be a smooth projective variety equipped with a Kähler form ω . Let L be a holomorphic line bundle on V and let h_L be a singular hermitian metric on L such that $\sqrt{-1}\Theta_{h_L}(L) \geq \varepsilon\omega$ for some $\varepsilon > 0$. Then*

$$H^i(V, \omega_V \otimes L \otimes \mathcal{J}(h_L)) = 0$$

for every $i > 0$, where ω_V is the canonical bundle of V and $\mathcal{J}(h_L)$ is the multiplier ideal sheaf of h_L .

A semiample line bundle is always semipositive. Thus we have Theorem B as a direct consequence of Theorem A. Theorem B is a generalization of Kollár's original injectivity theorem in [Kol1].

Theorem B (Kollár-type injectivity). *Let F be a holomorphic line bundle on a compact Kähler manifold X and let h be a singular hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Let N_1 and N_2 be semiample line bundles on X and let s be a nonzero global section of N_2 . Assume that $N_1^{\otimes a} \simeq N_2^{\otimes b}$ for some positive integers a and b . Then the map*

$$\times s : H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) \rightarrow H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2)$$

induced by $\otimes s$ is injective for every i , where ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h .

Remark 1.5. (1) If X is a smooth projective variety and (F, h) is trivial, then Theorem B recovers the original Kollár injectivity theorem (see [Kol1, Theorem 2.2]).

(2) For the proof of Theorem B, we may assume that $b = 1$, that is, $N_2 \simeq N_1^{\otimes a}$ by replacing s with s^b . We note that the composition

$$\begin{aligned} H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) &\xrightarrow{\times s} H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2) \\ &\xrightarrow{\times s^{b-1}} H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2^{\otimes b}) \end{aligned}$$

is the map $\times s^b$ induced by $\otimes s^b$.

Theorem C is a generalization of Kollár's torsion-free theorem and Theorem D is a generalization of Kollár's vanishing theorem (see [Kol1, Theorem 2.1]).

Theorem C (Kollár-type torsion-freeness). *Let $f : X \rightarrow Y$ be a surjective morphism from a compact Kähler manifold X onto a projective variety Y . Let F be a holomorphic line bundle on X and let h be a singular hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Then*

$$R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h))$$

is torsion-free for every i , where ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h .

Theorem D (Kollár-type vanishing theorem). *Let $f : X \rightarrow Y$ be a surjective morphism from a compact Kähler manifold X onto a projective variety Y . Let F be a holomorphic line bundle on X and let h be a singular hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Let N be a holomorphic line bundle on X . We assume that there exist positive integers a and b and an ample line bundle H on Y such that $N^{\otimes a} \simeq f^*H^{\otimes b}$. Then we obtain that*

$$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N)) = 0$$

for every $i > 0$ and j , where ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h .

Remark 1.6. (1) If X is a smooth projective variety and (F, h) is trivial, then Theorem C is nothing but Kollár's torsion-free theorem. Furthermore, if $N \simeq f^*H$, that is, $a = b = 1$, then Theorem D is the Kollár vanishing theorem. For the details, see [Kol1, Theorem 2.1].

(2) There exists a clever quick proof of Kollár's torsion-freeness by the theory of variations

of Hodge structure (see [Ar]).

(3) In [MaS6], the second author obtained a natural analytic generalization of Kollár's vanishing theorem, which contains Ohsawa's vanishing theorem (see [O1]) as a special case. The proof of this generalization depends on Takegoshi's theory of harmonic forms on complex manifolds with boundary (see [Ta]).

By combining Theorem D with the Castelnuovo–Mumford regularity, we can easily obtain Corollary 1.7, which is a complete generalization of [Hö, Lemma 3.35 and Remark 3.36]. The proof of [Hö, Lemma 3.35] depends on a generalization of the Ohsawa–Takegoshi L^2 extension theorem. We note that Höring claims the weak positivity of $f_*(\omega_{X/Y} \otimes F)$ under some extra assumptions by using [Hö, Lemma 3.35]. For the details, see [Hö, 3.H Multiplier ideals].

Corollary 1.7. *Let $f : X \rightarrow Y$ be a surjective morphism from a compact Kähler manifold X onto a projective variety Y . Let F be a holomorphic line bundle on X and let h be a singular hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Let H be an ample line bundle on Y such that $|H|$ is basepoint-free. Then*

$$R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}$$

is globally generated for every $i \geq 0$ and $m \geq \dim Y + 1$, where ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h .

As a direct consequence of Theorem D, we obtain Theorem E. For the definition of GV-sheaves in the sense of Pareschi and Popa, see Definition 1.8 below. For the details of GV-sheaves, we recommend the reader to see [Sc, Theorem 25.5 and Definition 26.3].

Theorem E (GV-sheaves). *Let $f : X \rightarrow A$ be a morphism from a compact Kähler manifold X to an Abelian variety A . Let F be a holomorphic line bundle on X and let h be a singular hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Then*

$$R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h))$$

is a GV-sheaf for every i , where ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h .

Definition 1.8 (GV-sheaves in the sense of Pareschi and Popa: [PP]). Let A be an Abelian variety. A coherent sheaf \mathcal{F} on A is said to be a GV-sheaf if

$$\mathrm{codim}_{\mathrm{Pic}^0(A)} \{L \in \mathrm{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes L) \neq 0\} \geq i$$

for every i .

The final one is a generalization of the generic vanishing theorem (see [GL], [Ha], [PP], and so on). The formulation of Theorem F is closer to [Ha] and [PP] than to the original generic vanishing theorem by Green and Lazarsfeld in [GL].

Theorem F (Generic vanishing theorem). *Let $f : X \rightarrow A$ be a morphism from a compact Kähler manifold X to an Abelian variety A . Let F be a holomorphic line bundle on X and let h be a singular hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Then*

$$\mathrm{codim}_{\mathrm{Pic}^0(A)} \{L \in \mathrm{Pic}^0(A) \mid H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^*L) \neq 0\} \geq i - (\dim X - \dim f(X))$$

for every $i \geq 0$, where ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h .

The main results explained above are closely related one another. The following proposition, which is also one of the main contributions in this paper, shows several relationships among them. By Proposition 1.9, we see that it is sufficient to prove Theorem A. The proof of Proposition 1.9 will be given in Section 4.

Proposition 1.9. *We have the following relationships among the above theorems.*

- (i) *Theorem A implies Theorem B.*
- (ii) *Theorem B is equivalent to Theorem C and Theorem D.*
- (iii) *Theorem D implies Theorem E.*
- (iv) *Theorem C and Theorem E imply Theorem F.*

A key ingredient of Proposition 1.9 is the following theorem, which can be seen as a Bertini-type theorem on the restriction of multiplier ideal sheaves to general members of free linear systems. Theorem 1.10 enables us to use the inductive argument on dimension, and thus it seems to be useful. We remark that \mathcal{G} in Theorem 1.10 is not always an intersection of countably many Zariski open sets (see Example 3.12). We will give a proof of Theorem 1.10 in Section 3, which is much harder than we expected.

Theorem 1.10 (Density of good divisors: Theorem 3.7). *Let X be a compact complex manifold, let Λ be a free linear system on X with $\dim \Lambda \geq 1$, and let φ be a quasi-plurisubharmonic function on X . We put*

$$\mathcal{G} := \{H \in \Lambda \mid H \text{ is smooth and } \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}.$$

Then \mathcal{G} is dense in Λ in the classical topology.

We note that all the results explained above hold even if we replace ω_X with $\omega_X \otimes E$, where E is any Nakano semipositive vector bundle on X . We will explain Theorem 1.11 in the final section: Section 6.

Theorem 1.11 (Twists by Nakano semipositive vector bundles). *Let E be a Nakano semipositive vector bundle on a compact Kähler manifold X . Then Theorems A, B, C, D, E, F, Theorem 1.4, Corollary 1.7, and Proposition 1.9 hold even when ω_X is replaced with $\omega_X \otimes E$.*

In this paper, we assume that all the varieties and manifolds are compact for simplicity. For some injectivity, torsion-free, and vanishing theorems for noncompact manifolds, see [Ta], [F5], [MaS5], and so on.

We summarize the contents of this paper. In Section 2, we recall some basic definitions and collect several preliminary lemmas. Section 3 is devoted to the proof of Theorem 1.10. Theorem 1.10 plays a crucial role in the proof of Proposition 1.9. In Section 4, we prove Proposition 1.9 and Corollary 1.7, and explain how to reduce Theorem 1.4 to Theorem A. By these results, we see that all we have to do is to establish Theorem A. In Section 5, which is the main part of this paper, we give a detailed proof of Theorem A. In the final section: Section 6, we prove Theorem 1.11. Precisely speaking, we explain how to modify the arguments used before for the proof of Theorem 1.11.

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2. PRELIMINARIES

Let us quickly recall the definition of singular hermitian metrics, (quasi-)plurisubharmonic functions, and Nadel's multiplier ideal sheaves. For the details, we recommend the reader to see [D3].

Definition 2.1 (Singular hermitian metrics and curvatures). Let F be a holomorphic line bundle on a complex manifold X . A singular hermitian metric on F is a metric h which is given in every trivialization $\theta : F|_\Omega \simeq \Omega \times \mathbb{C}$ by

$$|\xi|_h = |\theta(\xi)|e^{-\varphi} \text{ on } \Omega,$$

where ξ is a section of F on Ω and $\varphi \in L^1_{\text{loc}}(\Omega)$ is an arbitrary function. Here $L^1_{\text{loc}}(\Omega)$ is the space of locally integrable functions on Ω . We usually call φ the weight function of the metric with respect to the trivialization θ . The curvature of a singular hermitian metric h is defined by

$$\sqrt{-1}\Theta_h(F) := 2\sqrt{-1}\partial\bar{\partial}\varphi,$$

where φ is a weight function and $\sqrt{-1}\partial\bar{\partial}\varphi$ is taken in the sense of distributions. It is easy to see that the right hand side does not depend on the choice of trivializations.

The notion of multiplier ideal sheaves introduced by Nadel is very important in the recent developments of complex geometry and algebraic geometry.

Definition 2.2 ((Quasi-)plurisubharmonic functions and multiplier ideal sheaves). A function $u : \Omega \rightarrow [-\infty, \infty)$ defined on an open set $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic if

- u is upper semicontinuous, and
- for every complex line $L \subset \mathbb{C}^n$, the restriction $u|_{\Omega \cap L}$ to L is subharmonic on $\Omega \cap L$, that is, for every $a \in \Omega$ and $\xi \in \mathbb{C}^n$ satisfying $|\xi| < d(a, \Omega^c)$, the function u satisfies the mean inequality

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\xi) d\theta.$$

Let X be a complex manifold. A function $\varphi : X \rightarrow [-\infty, \infty)$ is said to be plurisubharmonic if there exists an open cover $X = \bigcup_{i \in I} U_i$ such that $\varphi|_{U_i}$ is plurisubharmonic on U_i ($\subset \mathbb{C}^n$) for every i . We can easily see that this definition is independent of the choice of open covers. A quasi-plurisubharmonic function is a function φ which is locally equal to the sum of a plurisubharmonic function and of a smooth function. If φ is a quasi-plurisubharmonic function on a complex manifold X , then the multiplier ideal sheaf $\mathcal{J}(\varphi) \subset \mathcal{O}_X$ is defined by

$$\Gamma(U, \mathcal{J}(\varphi)) := \{f \in \mathcal{O}_X(U) \mid |f|^2 e^{-2\varphi} \in L^1_{\text{loc}}(U)\}$$

for every open set $U \subset X$. Then it is known that $\mathcal{J}(\varphi)$ is a coherent ideal sheaf of \mathcal{O}_X (see, for example, [D3, (5.7) Lemma]).

Let S be a complex submanifold of X . Then the restriction $\mathcal{J}(\varphi)|_S$ of the multiplier ideal sheaf $\mathcal{J}(\varphi)$ to S is defined by the image of $\mathcal{J}(\varphi)$ under the natural surjective morphism $\mathcal{O}_X \rightarrow \mathcal{O}_S$, that is,

$$\mathcal{J}(\varphi)|_S = \mathcal{J}(\varphi) / \mathcal{J}(\varphi) \cap \mathcal{I}_S,$$

where \mathcal{I}_S is the defining ideal sheaf of S on X . We note that the restriction $\mathcal{J}(\varphi)|_S$ does not always coincide with $\mathcal{J}(\varphi) \otimes \mathcal{O}_S = \mathcal{J}(\varphi) / \mathcal{J}(\varphi) \mathcal{I}_S$.

We give the definition of $\mathcal{J}(h)$ in the theorems in Section 1.

Definition 2.3. Let F be a holomorphic line bundle on a complex manifold X and let h be a singular hermitian metric on F . We assume $\sqrt{-1}\Theta_h(F) \geq \gamma$ for some smooth $(1,1)$ -form γ on X . We fix a smooth hermitian metric h_∞ on F . Then we can write $h = h_\infty e^{-2\psi}$ for some $\psi \in L^1_{\text{loc}}(X)$. Then ψ coincides with a quasi-plurisubharmonic function φ on X almost everywhere. In this situation, we put $\mathcal{J}(h) := \mathcal{J}(\varphi)$. We note that $\mathcal{J}(h)$ is independent of h_∞ and is well-defined.

We close this section with the following lemmas, which will be used in the proof of Theorem A in Section 5.

Lemma 2.4. *Let ω and $\tilde{\omega}$ be positive $(1,1)$ -forms on an n -dimensional complex manifold with $\tilde{\omega} \geq \omega$. If u is an (n,q) -form, then $|u|_{\tilde{\omega}}^2 dV_{\tilde{\omega}} \leq |u|_{\omega}^2 dV_{\omega}$. Furthermore, if u is an $(n,0)$ -form, then $|u|_{\tilde{\omega}}^2 dV_{\tilde{\omega}} = |u|_{\omega}^2 dV_{\omega}$. Here $|u|_{\omega}$ (resp. $|u|_{\tilde{\omega}}$) is the pointwise norm of u with respect to ω (resp. $\tilde{\omega}$) and dV_{ω} (resp. $dV_{\tilde{\omega}}$) is the volume form defined by $dV_{\omega} := \omega^n/n!$ (resp. $dV_{\tilde{\omega}} := \tilde{\omega}^n/n!$).*

Proof. This lemma follows from simple computations. Thus we omit the proof. \square

Lemma 2.5. *Let $\varphi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded operator (continuous linear map) between Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. If $\{w_k\}_{k=1}^\infty$ weakly converges to w in \mathcal{H}_1 , then $\{\varphi(w_k)\}_{k=1}^\infty$ weakly converges to $\varphi(w)$.*

Proof. By taking the adjoint operator φ^* , for every $v \in \mathcal{H}_2$, we have

$$\langle \varphi(w_k), v \rangle_{\mathcal{H}_2} = \langle w_k, \varphi^*(v) \rangle_{\mathcal{H}_1} \rightarrow \langle w, \varphi^*(v) \rangle_{\mathcal{H}_1} = \langle \varphi(w), v \rangle_{\mathcal{H}_2}.$$

This completes the proof. \square

Lemma 2.6. *Let L be a closed subspace in a Hilbert space \mathcal{H} . Then L is closed with respect to the weak topology of \mathcal{H} , that is, if a sequence $\{w_k\}_{k=1}^\infty$ in L weakly converges to w , then the weak limit w belongs to L .*

Proof. By the orthogonal decomposition, there exists a closed subspace M such that $L = M^\perp$. Then it follows that $w \in M^\perp = L$ since we have $0 = \langle w_k, v \rangle_{\mathcal{H}} \rightarrow \langle w, v \rangle_{\mathcal{H}}$ for every $v \in M$. \square

3. RESTRICTION LEMMA

This section is devoted to the proof of Theorem 1.10 (see Theorem 3.7), which will play a crucial role in the proof of Proposition 1.9. Let us start with the following easy lemma. It is a direct consequence of the Ohsawa–Takegoshi L^2 extension theorem (see [OT, Theorem]).

Lemma 3.1 (Ohsawa–Takegoshi L^2 extension theorem). *Let X be a complex manifold and let φ be a quasi-plurisubharmonic function on X . We consider a sequence of hypersurfaces*

$$F_k \subset F_{k-1} \subset \cdots \subset F_1 \subset F_0 := X,$$

where F_i is a smooth hypersurface of F_{i-1} for every i . Then, by the Ohsawa–Takegoshi L^2 extension theorem, we obtain that

$$\mathcal{J}(\varphi|_{F_k}) \subset \mathcal{J}(\varphi|_{F_{k-1}})|_{F_k} \subset \cdots \subset \mathcal{J}(\varphi|_{F_1})|_{F_k} \subset \mathcal{J}(\varphi)|_{F_k}.$$

Proof. This is just a rephrasing of the Ohsawa–Takegoshi L^2 extension theorem (see [OT, Theorem]). \square

The following lemma is a key ingredient of the proof of Theorem 1.10 (see Theorem 3.7).

Lemma 3.2. *Let X and φ be as in Lemma 3.1. Let H_i be a Cartier divisor on X for $1 \leq i \leq k$. We assume the following condition:*

♠ *The divisor H_i is smooth for every $1 \leq i \leq k$ and $\sum_{i=1}^k H_i$ is a simple normal crossing divisor on X . Moreover, for every $1 \leq i_1 < i_2 < \dots < i_l \leq k$ and any $P \in H_{i_1} \cap H_{i_2} \cap \dots \cap H_{i_l}$, the set $\{f_{i_1}, f_{i_2}, \dots, f_{i_l}\}$ is a regular sequence for $\mathcal{O}_{X,P}/\mathcal{J}(\varphi)_P$, where f_i is a (local) defining equation of H_i for every i .*

We put $F_i := H_1 \cap H_2 \cap \dots \cap H_i$ for $1 \leq i \leq k$. We assume that the equality $\mathcal{J}(\varphi|_{F_k}) = \mathcal{J}(\varphi)|_{F_k}$ holds. Then $\mathcal{J}(\varphi|_{F_j}) = \mathcal{J}(\varphi)|_{F_j}$ holds in a neighborhood of F_k for every j .

We give a small remark on condition ♠ before we prove Lemma 3.2.

Remark 3.3. Condition ♠ in Lemma 3.2 does not depend on the order of $\{H_1, H_2, \dots, H_k\}$ (see, for example, [MaH, Theorem 16.3] and [AK, Chapter III, Corollary (3.5)]).

Let us prove Lemma 3.2.

Proof of Lemma 3.2. By condition ♠, the morphism γ in the following commutative diagram is injective.

$$(3.1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{J}(\varphi) \otimes \mathcal{O}_X(-H_1) & \xrightarrow{\alpha} & \mathcal{J}(\varphi) & \longrightarrow & \text{Coker } \alpha \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \beta \\ 0 & \longrightarrow & \mathcal{O}_X(-H_1) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{H_1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & (\mathcal{O}_X/\mathcal{J}(\varphi)) \otimes \mathcal{O}_X(-H_1) & \xrightarrow{\gamma} & \mathcal{O}_X/\mathcal{J}(\varphi) & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Therefore, β is also injective. This implies that $\text{Coker } \alpha = \mathcal{J}(\varphi)|_{H_1}$ by definition. Thus we obtain the following short exact sequence:

$$0 \rightarrow \mathcal{J}(\varphi) \otimes \mathcal{O}_X(-H_1) \rightarrow \mathcal{J}(\varphi) \rightarrow \mathcal{J}(\varphi)|_{H_1} \rightarrow 0.$$

We also obtain the following short exact sequence:

$$0 \rightarrow (\mathcal{O}_X/\mathcal{J}(\varphi)) \otimes \mathcal{O}_X(-H_1) \xrightarrow{\gamma} \mathcal{O}_X/\mathcal{J}(\varphi) \rightarrow \mathcal{O}_{H_1}/\mathcal{J}(\varphi)|_{H_1} \rightarrow 0$$

by the above big commutative diagram. Similarly, by condition ♠, we can inductively check that

$$(3.2) \quad 0 \rightarrow \mathcal{J}(\varphi)|_{F_i} \otimes \mathcal{O}_{F_i}(-H_{i+1}) \rightarrow \mathcal{J}(\varphi)|_{F_i} \rightarrow \mathcal{J}(\varphi)|_{F_{i+1}} \rightarrow 0$$

and

$$(3.3) \quad 0 \rightarrow (\mathcal{O}_{F_i}/\mathcal{J}(\varphi)|_{F_i}) \otimes \mathcal{O}_{F_i}(-H_{i+1}) \rightarrow \mathcal{O}_{F_i}/\mathcal{J}(\varphi)|_{F_i} \rightarrow \mathcal{O}_{F_{i+1}}/\mathcal{J}(\varphi)|_{F_{i+1}} \rightarrow 0$$

are exact for every $1 \leq i \leq k-1$. We consider the following big commutative diagram:

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
0 & \longrightarrow & \mathcal{J}(\varphi|_{F_i}) \otimes \mathcal{O}_{F_i}(-H_{i+1}) & \longrightarrow & \mathcal{J}(\varphi)|_{F_i} \otimes \mathcal{O}_{F_i}(-H_{i+1}) & \longrightarrow & \text{Coker } b_i \otimes \mathcal{O}_{F_i}(-H_{i+1}) \longrightarrow 0 \\
& & \downarrow a_i & & \downarrow & & \downarrow d_i \\
0 & \longrightarrow & \mathcal{J}(\varphi|_{F_i}) & \xrightarrow{b_i} & \mathcal{J}(\varphi)|_{F_i} & \longrightarrow & \text{Coker } b_i \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & \text{Coker } a_i & \xrightarrow{c_i} & \mathcal{J}(\varphi)|_{F_{i+1}} & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

for $0 \leq i \leq k-1$. The assumption $\mathcal{J}(\varphi|_{F_k}) = \mathcal{J}(\varphi)|_{F_k}$ implies that $\mathcal{J}(\varphi|_{F_{k-1}})|_{F_k} = \mathcal{J}(\varphi)|_{F_k}$ holds by $\mathcal{J}(\varphi|_{F_k}) \subset \mathcal{J}(\varphi|_{F_{k-1}})|_{F_k} \subset \cdots \subset \mathcal{J}(\varphi)|_{F_k}$ in Lemma 3.1. If $\mathcal{J}(\varphi|_{F_i})|_{F_{i+1}} = \mathcal{J}(\varphi)|_{F_{i+1}}$ in a neighborhood of F_k , then c_i is surjective in a neighborhood of F_k by the definition of $\mathcal{J}(\varphi|_{F_i})|_{F_{i+1}}$. Then d_i is also surjective in a neighborhood of F_k by the above big commutative diagram. By Nakayama's lemma, $\text{Coker } b_i$ is zero in a neighborhood of F_k . This implies that $\mathcal{J}(\varphi|_{F_i}) = \mathcal{J}(\varphi)|_{F_i}$ in a neighborhood of F_k . Thus we obtain that $\mathcal{J}(\varphi|_{F_{i-1}})|_{F_i} = \mathcal{J}(\varphi)|_{F_i}$ in a neighborhood of F_k since we have $\mathcal{J}(\varphi|_{F_i}) \subset \mathcal{J}(\varphi|_{F_{i-1}})|_{F_i} \subset \mathcal{J}(\varphi)|_{F_i}$ by Lemma 3.1. By repeating this argument, we see that $\mathcal{J}(\varphi|_{F_j}) = \mathcal{J}(\varphi)|_{F_j}$ in a neighborhood of F_k for every j . This is the desired property. \square

We give a very important remark on condition \spadesuit in Lemma 3.2.

Remark 3.4. Condition \spadesuit is equivalent to the following condition:

- The divisor H_i is smooth for every $1 \leq i \leq k$ and $\sum_{i=1}^k H_i$ is a simple normal crossing divisor on X . Moreover, for every $1 \leq i_1 < i_2 < \cdots < i_{l-1} < i_l \leq k$, the divisor H_{i_l} contains no associated primes of $\mathcal{O}_X/\mathcal{J}(\varphi)$ and $\mathcal{O}_{H_{i_1} \cap \cdots \cap H_{i_{l-1}}}/\mathcal{J}(\varphi)|_{H_{i_1} \cap \cdots \cap H_{i_{l-1}}}$.

In order to understand this condition, (3.4) below may be helpful. We put $H_{i_1 \cdots i_m} := H_{i_1} \cap \cdots \cap H_{i_m}$ for every $1 \leq i_1 < \cdots < i_m \leq k$. Then we can inductively check that

$$0 \rightarrow \mathcal{J}(\varphi)|_{H_{i_1 \cdots i_{l-1}}} \otimes \mathcal{O}_{H_{i_1 \cdots i_{l-1}}}(-H_{i_l}) \rightarrow \mathcal{J}(\varphi)|_{H_{i_1 \cdots i_{l-1}}} \rightarrow \mathcal{J}(\varphi)|_{H_{i_1 \cdots i_l}} \rightarrow 0$$

is exact (see (3.2) in the proof of Lemma 3.2) and that

$$\begin{aligned}
(3.4) \quad 0 & \rightarrow \left(\mathcal{O}_{H_{i_1 \cdots i_{l-1}}}/\mathcal{J}(\varphi)|_{H_{i_1 \cdots i_{l-1}}} \right) \otimes \mathcal{O}_{H_{i_1 \cdots i_{l-1}}}(-H_{i_l}) \\
& \rightarrow \mathcal{O}_{H_{i_1 \cdots i_{l-1}}}/\mathcal{J}(\varphi)|_{H_{i_1 \cdots i_{l-1}}} \rightarrow \mathcal{O}_{H_{i_1 \cdots i_l}}/\mathcal{J}(\varphi)|_{H_{i_1 \cdots i_l}} \rightarrow 0
\end{aligned}$$

is also exact (see (3.3) in the proof of Lemma 3.2) as in the proof of Lemma 3.2.

By Remark 3.4, the following lemmas are almost obvious.

Lemma 3.5. Assume that $\{H_1, \dots, H_m\}$ satisfies condition \spadesuit in Lemma 3.2. Let H_{m+1} be a smooth Cartier divisor on X such that $\sum_{i=1}^{m+1} H_i$ is a simple normal crossing divisor on X and that H_{m+1} contains no associated primes of

$$\mathcal{O}_X/\mathcal{J}(\varphi) \quad \text{and} \quad \mathcal{O}_{H_{i_1} \cap \cdots \cap H_{i_l}}/\mathcal{J}(\varphi)|_{H_{i_1} \cap \cdots \cap H_{i_l}}$$

for every $1 \leq i_1 < \cdots < i_l \leq m$. Then $\{H_1, \dots, H_m, H_{m+1}\}$ also satisfies \spadesuit .

Proof. This is obvious by Remark 3.4. \square

Lemma 3.6. *Let Λ_0 be a (not necessarily free) linear system on X with $\dim \Lambda_0 \geq 1$. Assume that $\{H_1, \dots, H_m\}$ satisfies condition \spadesuit in Lemma 3.2. We put*

$$\mathcal{G}_0 := \{D \in \Lambda_0 \mid \{H_1, \dots, H_m, D\} \text{ satisfies } \spadesuit\}.$$

Then \mathcal{G}_0 is Zariski open in Λ_0 . In particular, if \mathcal{G}_0 is not empty, then it is a dense Zariski open set of Λ_0 .

Moreover, we assume that there exists $D_0 \in \mathcal{G}_0$ such that $\mathcal{J}(\varphi|_V) = \mathcal{J}(\varphi)|_V$, where V is an irreducible component of $H_1 \cap \cdots \cap H_m \cap D_0$. Let D be a member of \mathcal{G}_0 such that V is an irreducible component of $H_1 \cap \cdots \cap H_m \cap D$. Then $\mathcal{J}(\varphi|_D) = \mathcal{J}(\varphi)|_D$ holds in a neighborhood of V .

Proof. By Remark 3.4 and Lemma 3.5, it is obvious that \mathcal{G}_0 is Zariski open. By Lemma 3.2, the equality $\mathcal{J}(\varphi|_D) = \mathcal{J}(\varphi)|_D$ holds in a neighborhood of V if $D \in \mathcal{G}_0$ and V is an irreducible component of $H_1 \cap \cdots \cap H_m \cap D$. \square

The following theorem (see Theorem 1.10) is one of the key results of this paper. This theorem is missing in [F4].

Theorem 3.7 (Density of good divisors: Theorem 1.10). *Let X be a compact complex manifold, let Λ be a free linear system on X with $\dim \Lambda \geq 1$, and let φ be a quasi-plurisubharmonic function on X . We put*

$$\mathcal{G} := \{H \in \Lambda \mid H \text{ is smooth and } \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}.$$

Then \mathcal{G} is dense in Λ in the classical topology.

Proof. We divide the proof into several small steps.

Step 0 (Idea of the proof). In this step, we will explain the idea of the proof.

If $\dim \Lambda = 1$, that is, Λ is a pencil, then we obtain a morphism $f := \Phi_\Lambda : X \rightarrow \mathbb{P}^1$. By Fubini's theorem, we have $\mathcal{J}(\varphi|_{f^{-1}(P)}) \supset \mathcal{J}(\varphi)|_{f^{-1}(P)}$ for almost all $P \in \mathbb{P}^1$. On the other hand $\mathcal{J}(\varphi|_{f^{-1}(P)}) \subset \mathcal{J}(\varphi)|_{f^{-1}(P)}$ always holds by the Ohsawa–Takegoshi L^2 extension theorem when f^*P is a smooth divisor on X . Therefore $\mathcal{J}(\varphi|_{f^{-1}(P)}) = \mathcal{J}(\varphi)|_{f^{-1}(P)}$ holds for almost all $P \in \mathbb{P}^1$. This is the desired statement when $\dim \Lambda = 1$. In general $H_1 \cap H_2 \neq \emptyset$ for two general members H_1 and H_2 of Λ . For this reason, we choose H_1 and H_2 suitably (see Step 4 and Step 5), take the blow-up $Z \rightarrow X$ along $H_1 \cap H_2$, and reduce the problem to the pencil case (see Step 6).

Step 1. Let H be a smooth member of Λ . Then we obtain that $\mathcal{J}(\varphi|_H) \subset \mathcal{J}(\varphi)|_H$ by the Ohsawa–Takegoshi L^2 extension theorem (see Lemma 3.1).

Step 2. If $\varphi \equiv -\infty$ on X , then we obtain that $\varphi|_H \equiv -\infty$, $\mathcal{J}(\varphi|_H) = 0$, and $\mathcal{J}(\varphi) = 0$. Therefore, we have $\mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H$ for any smooth member H of Λ .

Therefore, from now on, we assume that $\varphi \not\equiv -\infty$ on X . We put $f := \Phi_\Lambda : X \rightarrow Y := f(X) \subset \mathbb{P}^N$. Of course, $N = \dim \Lambda$.

Step 3. In this step, we will prove that \mathcal{G} is dense in Λ in the classical topology when $\dim Y = 1$.

We can take a nonempty Zariski open set U of Y such that U is smooth and that f is smooth over U . Then, by Fubini's theorem, we see that $\mathcal{J}(\varphi|_{f^{-1}(P)}) \supset \mathcal{J}(\varphi)|_{f^{-1}(P)}$ for almost all $P \in U$. By Step 1, we have $\mathcal{J}(\varphi|_{f^{-1}(P)}) = \mathcal{J}(\varphi)|_{f^{-1}(P)}$ for almost all $P \in U$. Then we can see that \mathcal{G} is dense in Λ in the classical topology by Lemma 3.8 below.

Lemma 3.8. *Let C be an irreducible curve in \mathbb{P}^M . Let \mathcal{N} be a subset of C such that \mathcal{N} has Lebesgue measure zero in the regular locus of C . Then*

$$\{H \in |\mathcal{O}_{\mathbb{P}^M}(1)| \mid C \cap H \text{ is smooth and } \mathcal{N} \cap H = \emptyset\}$$

is dense in $|\mathcal{O}_{\mathbb{P}^M}(1)| \simeq \mathbb{P}^M$ in the classical topology.

Proof of Lemma 3.8. We take a general member $H^\dagger \in |\mathcal{O}_{\mathbb{P}^M}(1)|$. Then $C \cap H^\dagger$ is smooth by Bertini's theorem. If H^\dagger moves in $|\mathcal{O}_{\mathbb{P}^M}(1)| \simeq \mathbb{P}^M$ holomorphically in the general direction, then every point of $C \cap H^\dagger$ moves nontrivially and holomorphically. Therefore, we can find $\{H_i\}_{i=1}^\infty$ such that $C \cap H_i$ is smooth and $\mathcal{N} \cap H_i = \emptyset$ for every i and that $\lim_{i \rightarrow \infty} H_i = H^\dagger$ in $|\mathcal{O}_{\mathbb{P}^M}(1)| \simeq \mathbb{P}^M$ in the classical topology. This implies the desired property. \square

Step 4. In this step, we will prove the following preparatory lemma.

Lemma 3.9. *Let D_1 and D_2 be two general members of Λ such that $\{D_1, D_2\}$ satisfies \spadesuit in Lemma 3.2. Let \mathcal{P}_0 be the pencil spanned by D_1 and D_2 . Then, for almost all $D \in \mathcal{P}_0$, D is smooth, $\{D\}$ satisfies \spadesuit , and $\mathcal{J}(\varphi|_D) = \mathcal{J}(\varphi)|_D$ holds outside $D_1 \cap D_2$.*

Proof of Lemma 3.9. Let $p : Z \rightarrow X$ be the blow-up along $D_1 \cap D_2$. Then we have the following commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ & \searrow q & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

where $X \dashrightarrow \mathbb{P}^1$ is the meromorphic map corresponding to \mathcal{P}_0 . By applying Fubini's theorem to $q : Z \rightarrow \mathbb{P}^1$, we obtain that $\mathcal{J}(p^*\varphi|_{q^{-1}(Q)}) = \mathcal{J}(p^*\varphi)|_{q^{-1}(Q)}$ for almost all $Q \in \mathbb{P}^1$. By Lemma 3.6, $\{D\}$ satisfies \spadesuit for almost all $D \in \mathcal{P}_0$. Since p is an isomorphism outside $D_1 \cap D_2$, we have the desired properties. \square

Step 5. In this step, we will find a smooth member H of Λ such that $\mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H$ and that $\{H\}$ satisfies \spadesuit .

Let $f : X \rightarrow Y \subset \mathbb{P}^N$ be as above. We will use the induction on the dimension $N = \dim \Lambda$. If $N = 1$, that is, Λ is a pencil, then $Y = \mathbb{P}^1$. In this case, we see that \mathcal{G} is dense in Λ in the classical topology by Step 3. In particular, we have a smooth member H of Λ such that $\mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H$ and that $\{H\}$ satisfies \spadesuit . From now on, we assume that $\dim \Lambda \geq 2$ and that the statement of Theorem 3.7 holds for lower dimensional free linear systems. We take two general hyperplanes B_1 and B_2 of \mathbb{P}^N . We put $D_1 := f^*B_1$ and $D_2 := f^*B_2$. By Lemma 3.9, we can take a hyperplane A_1 of \mathbb{P}^N such that $X_1 := f^*A_1$ is smooth, $\{X_1\}$ satisfies \spadesuit , and $\mathcal{J}(\varphi|_{X_1}) = \mathcal{J}(\varphi)|_{X_1}$ outside $D_1 \cap D_2$. By construction, we have $\dim \Lambda|_{X_1} = \dim \Lambda - 1$. Thus we see that

$$\{H \in \Lambda \mid X_1 \cap H \text{ is smooth and } \mathcal{J}(\varphi|_{X_1 \cap H}) = \mathcal{J}(\varphi|_{X_1})|_{X_1 \cap H}\}$$

is dense in Λ in the classical topology by the induction hypothesis. We put $l := \dim Y \geq 1$. Then we can take general hyperplanes A_2, A_3, \dots, A_l of \mathbb{P}^N such that $\dim(A_1 \cap \dots \cap A_l \cap$

$Y) = 0$ and that $f^{-1}(Q)$ is smooth and

$$(3.5) \quad \mathcal{J}(\varphi|_{f^{-1}(Q)}) = \mathcal{J}(\varphi|_{X_1})|_{f^{-1}(Q)}$$

for every $Q \in A_1 \cap \cdots \cap A_l \cap Y$ by using the induction hypothesis repeatedly. Without loss of generality, we may assume that $f^{-1}(Q) \cap D_1 \cap D_2 = \emptyset$ for every $Q \in A_1 \cap \cdots \cap A_l \cap Y$. Since

$$\mathcal{J}(\varphi|_{X_1}) = \mathcal{J}(\varphi)|_{X_1}$$

outside $D_1 \cap D_2$,

$$(3.6) \quad \mathcal{J}(\varphi|_{X_1})|_{f^{-1}(Q)} = \mathcal{J}(\varphi)|_{f^{-1}(Q)}$$

holds for every $Q \in A_1 \cap \cdots \cap A_l \cap Y$. Therefore, we have

$$\mathcal{J}(\varphi|_{f^{-1}(Q)}) = \mathcal{J}(\varphi|_{X_1})|_{f^{-1}(Q)} = \mathcal{J}(\varphi)|_{f^{-1}(Q)}$$

for every $Q \in A_1 \cap \cdots \cap A_l \cap Y$ by (3.5) and (3.6). Of course, we may assume that $\{X_1 = f^*A_1, f^*A_2, \dots, f^*A_l\}$ satisfies \spadesuit . We take one point P of $A_1 \cap \cdots \cap A_l \cap Y$ and fix A_2, \dots, A_l . By applying Lemma 3.6 to the linear system

$$\Lambda_0 := \{D \in \Lambda \mid f^{-1}(P) \subset D\},$$

we know that

$$\mathcal{G}_0 := \{D \in \Lambda_0 \mid \{D, f^*A_2, \dots, f^*A_l\} \text{ satisfies } \spadesuit\}$$

is Zariski open in Λ_0 . Note that \mathcal{G}_0 is nonempty by $X_1 = f^*A_1 \in \mathcal{G}_0$. By the latter conclusion of Lemma 3.6, we have:

Lemma 3.10. *Let A_g be a general hyperplane of \mathbb{P}^N passing through P . We put $X_g := f^*A_g$. Then $\mathcal{J}(\varphi|_{X_g}) = \mathcal{J}(\varphi)|_{X_g}$ holds in a neighborhood of $f^{-1}(P)$.*

Let $\pi : X' \rightarrow X$ be the blow-up along $f^{-1}(P)$ and let $\text{Bl}_P(\mathbb{P}^N) \rightarrow \mathbb{P}^N$ be the blow-up of \mathbb{P}^N at P . Then we obtain the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \alpha \downarrow & & \downarrow f \\ \text{Bl}_P(\mathbb{P}^N) & \longrightarrow & \mathbb{P}^N \\ \beta \downarrow & \nearrow \gamma & \\ \mathbb{P}^{N-1} & & \end{array}$$

Note that α is naturally induced by $f : X \rightarrow \mathbb{P}^N$ and that $\gamma : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ is the linear projection from $P \in \mathbb{P}^N$. We put $f' := \beta \circ \alpha$ and $Y' := f'(X')$. By applying the induction hypothesis to $f' : X' \rightarrow Y' \subset \mathbb{P}^{N-1}$, we can take a general hyperplane A of \mathbb{P}^{N-1} such that f'^*A is smooth and that

$$(3.7) \quad \mathcal{J}(\pi^*\varphi|_{f'^{-1}(A)}) = \mathcal{J}(\pi^*\varphi)|_{f'^{-1}(A)}.$$

Let A_0 be the hyperplane of \mathbb{P}^N spanned by P and A . Then we can see that

$$(3.8) \quad \{f^*A_2, \dots, f^*A_l, H := f^*A_0\}$$

satisfies \spadesuit because A is a general hyperplane of \mathbb{P}^{N-1} . By Lemma 3.10 and (3.7), we see that $\mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H$. We note that $\{H\}$ satisfies \spadesuit by (3.8). Therefore, this H has the desired properties.

Step 6. In this final step, we will prove that \mathcal{G} is dense in Λ in the classical topology.

We will use the induction on $\dim X$. If $\dim X = 1$, then $\dim Y = 1$. Therefore, by Step 3, we see that \mathcal{G} is dense in Λ in the classical topology. So, we assume that $\dim X \geq 2$. If $\dim \Lambda = 1$, then \mathcal{G} is dense by Step 3. Thus we may assume that $\dim \Lambda \geq 2$. By Step 5, we can take a smooth member H_0 of Λ such that $\mathcal{J}(\varphi|_{H_0}) = \mathcal{J}(\varphi)|_{H_0}$ and that $\{H_0\}$ satisfies \spadesuit . By applying the induction hypothesis to $\Lambda|_{H_0}$, we see that

$$\mathcal{G}' := \{H' \in \Lambda \mid H_0 \cap H' \text{ is smooth and } \mathcal{J}(\varphi|_{H_0 \cap H'}) = \mathcal{J}(\varphi|_{H_0})|_{H_0 \cap H'}\}$$

is dense in Λ in the classical topology. Since Λ is a free linear system, we know that

$$\{H' \in \Lambda \mid \{H_0, H'\} \text{ satisfies } \spadesuit\}$$

is a nonempty Zariski open set in Λ . Therefore,

$$\mathcal{G}'' := \{H' \in \mathcal{G}' \mid \{H_0, H'\} \text{ satisfies } \spadesuit\}.$$

is also dense in Λ in the classical topology. We note that

$$\mathcal{J}(\varphi|_{H_0 \cap H'}) = \mathcal{J}(\varphi|_{H_0})|_{H_0 \cap H'} = \mathcal{J}(\varphi)|_{H_0 \cap H'}$$

for every $H' \in \mathcal{G}'$ since $\mathcal{J}(\varphi|_{H_0}) = \mathcal{J}(\varphi)|_{H_0}$. Therefore, we obtain that

$$(3.9) \quad \mathcal{J}(\varphi|_{H_0 \cap H'}) = \mathcal{J}(\varphi|_{H'})|_{H_0 \cap H'} = \mathcal{J}(\varphi)|_{H_0 \cap H'}$$

for every $H' \in \mathcal{G}'$. By the latter conclusion of Lemma 3.6, (3.9) implies that $\mathcal{J}(\varphi|_{H'}) = \mathcal{J}(\varphi)|_{H'}$ in a neighborhood of $H_0 \cap H'$ in H' for every $H' \in \mathcal{G}''$. We consider the pencil $\mathcal{P}_{H'}$ spanned by H_0 and $H' \in \mathcal{G}''$, that is, the sublinear system of Λ spanned by H_0 and H' . By taking the blow-up of X along $H_0 \cap H'$ and applying the arguments in Step 3 and Lemma 3.9, we see that almost all members of $\mathcal{P}_{H'}$ are contained in \mathcal{G} (see also Lemma 3.6). By this observation, we obtain that \mathcal{G} is dense in Λ in the classical topology.

Thus we obtain the desired statement. \square

The following example shows that \mathcal{G} in Theorem 1.10 (Theorem 3.7) is not always Zariski open in Λ .

Example 3.11. We put

$$\psi(z) := \sum_{k=1}^{\infty} 2^{-k} \log \left| z - \frac{1}{k} \right|$$

for $z \in \mathbb{C}$. Then it is easy to see that $\psi(z)$ is smooth for $|z| \geq 2$. By using a suitable partition of unity, we can construct a function $\varphi(z)$ on \mathbb{P}^1 such that $\varphi(z) = \psi(z)$ for $|z| \leq 3$ and that $\varphi(z)$ is smooth for $|z| \geq 2$ on \mathbb{P}^1 . We can see that φ is a quasi-plurisubharmonic function on \mathbb{P}^1 . Since the Lelong number $\nu(\varphi, 1/n)$ of φ at $1/n$ is 2^{-n} for every positive integer n , we see that $\mathcal{J}(\varphi) = \mathcal{O}_{\mathbb{P}^1}$ by Skoda's theorem (see, for example, [D3, (5.6) Lemma]). Therefore $\mathcal{J}(\varphi)|_P = \mathcal{O}_P$ for every $P \in \mathbb{P}^1$. On the other hand, we have $\varphi(1/n) = -\infty$ for every positive integer n . If $P = 1/n$ for some positive integer n , then $\mathcal{J}(\varphi|_P) = 0$. Thus,

$$\mathcal{G} := \{H \in |\mathcal{O}_{\mathbb{P}^1}(1)| \mid \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}$$

is not a Zariski open set of $|\mathcal{O}_{\mathbb{P}^1}(1)|$ ($\simeq \mathbb{P}^1$).

The authors learned the following example from Toshiyuki Sugawa, which shows that \mathcal{G} in Theorem 1.10 (Theorem 3.7) is not always an intersection of countably many nonempty Zariski open sets of Λ .

Example 3.12. We put $K := \{z \in \mathbb{C} \mid |z| \leq 1\}$. Let $\{w_n\}_{n=1}^\infty$ be a countable dense subset of K and let $\{a_n\}_{n=1}^\infty$ be positive real numbers such that $\sum_{n=1}^\infty a_n < \infty$. We put

$$\psi(z) := \sum_{n=1}^\infty a_n \log |z - w_n|$$

for $z \in \mathbb{C}$. Then we see that

- ψ is subharmonic on \mathbb{C} and $\psi \not\equiv -\infty$,
- $\psi = -\infty$ on an uncountable dense subset of K , and
- ψ is discontinuous almost everywhere on K .

For the details, see [Ra, Theorem 2.5.4]. By using a suitable partition of unity, we can construct a function $\varphi(z)$ on \mathbb{P}^1 such that $\varphi(z) = \psi(z)$ for $|z| \leq 3$ and that $\varphi(z)$ is smooth for $|z| \geq 2$ on \mathbb{P}^1 . Then we can see that φ is a quasi-plurisubharmonic function on \mathbb{P}^1 . In this case,

$$\mathcal{G} := \{H \in |\mathcal{O}_{\mathbb{P}^1}(1)| \mid \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}$$

can not be written as an intersection of countably many nonempty Zariski open sets of $|\mathcal{O}_{\mathbb{P}^1}(1)|$. Of course, we can easily see that \mathcal{G} is dense in $|\mathcal{O}_{\mathbb{P}^1}(1)|$ in the classical topology.

As a direct consequence of Theorem 3.7, we have:

Corollary 3.13 (Generic restriction theorem). *Let X be a compact complex manifold and let φ be a quasi-plurisubharmonic function on X . Let Λ be a free linear system on X with $\dim \Lambda \geq 1$. We put*

$$\mathcal{H} := \{H \in \Lambda \mid H \text{ contains no associated primes of } \mathcal{O}_X/\mathcal{J}(\varphi)\},$$

where

$$\mathcal{G} := \{H \in \Lambda \mid H \text{ is smooth and } \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}$$

as in Theorem 3.7. Then \mathcal{H} is dense in Λ in the classical topology. Moreover, the following short sequence

$$(3.10) \quad 0 \rightarrow \mathcal{J}(\varphi) \otimes \mathcal{O}_X(-H) \rightarrow \mathcal{J}(\varphi) \rightarrow \mathcal{J}(\varphi|_H) \rightarrow 0$$

is exact for any member H of \mathcal{H} .

Proof. It is easy to see that

$$\{H \in \Lambda \mid H \text{ contains no associated primes of } \mathcal{O}_X/\mathcal{J}(\varphi)\}$$

is a nonempty Zariski open set of Λ because Λ is a free linear system on X . Therefore \mathcal{H} is dense in Λ in the classical topology by Theorem 3.7 (see Theorem 1.10).

Let H be a member of \mathcal{H} . Then we obtain the following commutative diagram (see also (3.1)).

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{J}(\varphi) \otimes \mathcal{O}_X(-H) & \xrightarrow{\alpha} & \mathcal{J}(\varphi) & \longrightarrow & \text{Coker } \alpha \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X(-H) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_H \longrightarrow 0 \end{array}$$

As in the proof of Lemma 3.2, we obtain $\text{Coker } \alpha = \mathcal{J}(\varphi)|_H$. Since $H \in \mathcal{H} \subset \mathcal{G}$, we have $\mathcal{J}(\varphi)|_H = \mathcal{J}(\varphi|_H)$. Therefore, we obtain the desired short exact sequence (3.10). \square

We will use Corollary 3.13 in Step 3 in the proof of Proposition 1.9 (see Section 4). We close this section with a remark on the multiplier ideal sheaves associated to effective \mathbb{Q} -divisors on smooth projective varieties.

Remark 3.14 (Multiplier ideal sheaves for effective \mathbb{Q} -divisors). Let X be a smooth projective variety and let D be an effective \mathbb{Q} -divisor on X . Let S be a smooth hypersurface in X . We assume that S is not contained in any component of D . Then we obtain the following short exact sequence:

$$(3.11) \quad 0 \rightarrow \mathcal{J}(X, D) \otimes \mathcal{O}_X(-S) \rightarrow \text{Adj}_S(X, D) \rightarrow \mathcal{J}(S, D|_S) \rightarrow 0,$$

where $\mathcal{J}(X, D)$ (resp. $\mathcal{J}(S, D|_S)$) is the multiplier ideal sheaf associated to D (resp. $D|_S$). Note that $\text{Adj}_S(X, D)$ is the adjoint ideal of D along S (see, for example, [L3, Theorem 3.3]). If S is in general position with respect to D , then we can easily see that $\text{Adj}_S(X, D)$ coincides with $\mathcal{J}(X, D)$. Let H be a general member of a free linear system Λ with $\dim \Lambda \geq 1$. Then we can easily see that

$$(3.12) \quad \mathcal{J}(H, D|_H) = \mathcal{J}(X, D)|_H$$

holds by the definition of the multiplier ideal sheaves for effective \mathbb{Q} -divisors (see, for example, [L2, Example 9.5.9]).

By this observation, if X is a smooth projective variety and φ is a quasi-plurisubharmonic function associated to an effective \mathbb{Q} -divisor D on X , then \mathcal{G} in Theorem 3.7 (see Theorem 1.10) and \mathcal{H} in Corollary 3.13 are dense Zariski open in Λ by (3.12). Moreover, we can easily check that (3.10) in Corollary 3.13 holds for general members H of Λ by (3.11).

4. PROOF OF PROPOSITION 1.9

In this section, we prove Proposition 1.9 and explain how to reduce Corollary 1.7 and Theorem 1.4 to Theorem D and Theorem A respectively.

Proof of Proposition 1.9. Our proof of Proposition 1.9 consists of the following six steps:

Step 1 (Theorem A \implies Theorem B). Since N_1 is semiample, we can take a smooth hermitian metric h_1 on N_1 such that $\sqrt{-1}\Theta_{h_1}(N_1) \geq 0$. We put $h_2 := h_1^{b/a}$. Then

$$\sqrt{-1}(\Theta_{hh_1}(F \otimes N_1) - t\Theta_{h_2}(N_2)) \geq 0$$

for $0 < t \ll 1$. We note that $\mathcal{J}(hh_1) = \mathcal{J}(h)$ because h_1 is smooth. Therefore, by Theorem A, we obtain the injectivity in Theorem B.

Step 2 (Theorem B \implies Theorem C). We assume that $R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h))$ has a torsion subsheaf. Then we can find a very ample line bundle H on Y and $0 \neq t \in H^0(Y, H)$ such that

$$\alpha : R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \rightarrow R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H$$

induced by $\otimes t$ is not injective. We take a sufficiently large positive integer m such that $\text{Ker } \alpha \otimes H^{\otimes m}$ is generated by global sections. Then we have $H^0(Y, \text{Ker } \alpha \otimes H^{\otimes m}) \neq 0$. Without loss of generality, by making m sufficiently large, we may further assume that

$$(4.1) \quad H^p(Y, R^q f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}) = 0$$

and

$$(4.2) \quad H^p(Y, R^q f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m+1}) = 0$$

for every $p > 0$ and q by the Serre vanishing theorem. By construction,

$$(4.3) \quad H^0(Y, R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}) \rightarrow H^0(Y, R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m+1})$$

induced by α is not injective. Thus, by (4.1), (4.2), and (4.3), we see that

$$H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^* H^{\otimes m}) \rightarrow H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^* H^{\otimes m+1})$$

induced by $\otimes f^* t$ is not injective. This contradicts Theorem B. Therefore $R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h))$ is torsion-free.

Step 3 (Theorem B \implies Theorem D). We use the induction on $\dim Y$. If $\dim Y = 0$, then the statement is obvious. We take a sufficiently large positive integer m and a general divisor $B \in |H^{\otimes m}|$ such that $D := f^{-1}(B)$ is smooth, contains no associated primes of $\mathcal{O}_X/\mathcal{J}(h)$, and satisfies $\mathcal{J}(h|_D) = \mathcal{J}(h)|_D$ by Theorem 3.7 (see Theorem 1.10) and Corollary 3.13. By the Serre vanishing theorem, we may further assume that

$$(4.4) \quad H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N) \otimes H^{\otimes m}) = 0$$

for every $i > 0$ and j . By Corollary 3.13 and adjunction, we have the following short exact sequence:

$$(4.5) \quad \begin{aligned} 0 &\rightarrow \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N \rightarrow \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N \otimes f^* H^{\otimes m} \\ &\rightarrow \omega_D \otimes F|_D \otimes \mathcal{J}(h|_D) \otimes N|_D \rightarrow 0. \end{aligned}$$

By (4.5), we have

$$\begin{aligned} 0 &\rightarrow R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N) \rightarrow R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N) \otimes H^{\otimes m} \\ &\rightarrow R^j f_*(\omega_D \otimes F|_D \otimes \mathcal{J}(h|_D) \otimes N|_D) \rightarrow 0 \end{aligned}$$

for every j since B is a general member of $|H^{\otimes m}|$. By using the long exact sequence and the induction on $\dim Y$, we obtain

$$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N)) = H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N) \otimes H^{\otimes m})$$

for every $i \geq 2$ and j . Thus we have

$$(4.6) \quad H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N)) = 0$$

for every $i \geq 2$ and j by (4.4). By Leray's spectral sequence and (4.4) and (4.6), we have the following commutative diagram:

$$\begin{array}{ccc} H^1(Y, \mathcal{S}^j) & \hookrightarrow & H^{j+1}(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N) \\ \alpha \downarrow & & \downarrow \beta \\ H^1(Y, \mathcal{S}^j \otimes H^{\otimes m}) & \hookrightarrow & H^{j+1}(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N \otimes f^* H^{\otimes m}) \end{array}$$

for every j , where \mathcal{S}^j stands for $R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N)$. Since β is injective by Theorem B, we obtain that α is also injective. By (4.4), we have

$$H^1(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N) \otimes H^{\otimes m}) = 0$$

for every j . Therefore, we have $H^1(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N)) = 0$ for every j . Thus we obtain the desired vanishing theorem in Theorem D.

Step 4 (Theorems C and D \implies Theorem B). By replacing s and N_2 with $s^{\otimes m}$ and $N_2^{\otimes m}$ for some positive integer m (see also Remark 1.5), we may assume that $|N_2|$ is basepoint-free. We consider

$$f := \Phi_{|N_2|} : X \rightarrow Y.$$

Then $N_2 \simeq f^*H$ for some ample line bundle H on Y and $s = f^*t$ for some $t \in H^0(Y, H)$. We take a smooth hermitian metric h_1 on N_1 such that $\sqrt{-1}\Theta_{h_1}(N_1) \geq 0$. Then $\sqrt{-1}\Theta_{hh_1}(F \otimes N_1) \geq 0$ and $\mathcal{J}(hh_1) = \mathcal{J}(h)$. By Theorem C, we obtain that

$$R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1)$$

is torsion-free for every i . Therefore, the map

$$R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) \rightarrow R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) \otimes H$$

induced by $\otimes t$ is injective for every i . By $N_2 \simeq f^*H$, we see that

$$(4.7) \quad H^0(Y, R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1)) \rightarrow H^0(Y, R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2))$$

induced by $\otimes t$ is injective for every i . By Theorem D, (4.7) implies that

$$H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) \rightarrow H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2)$$

induced by $\otimes s$ is injective for every i .

Step 5 (Theorem D \implies Theorem E). The following lemma implies that $R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))$ is a GV-sheaf by [Sc, Theorem 25.5] (see also [Ha] and [PP]). For simplicity, we put $\mathcal{F}^j := R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))$ for every j .

Lemma 4.1. *For every finite étale morphism $p : B \rightarrow A$ of Abelian varieties and every ample line bundle H on B , we have*

$$(4.8) \quad H^i(B, H \otimes p^* \mathcal{F}^j) = 0$$

for every $i > 0$ and j .

Proof of Lemma 4.1. We put $Z := B \times_A X$. Then we have the following commutative diagram.

$$(4.9) \quad \begin{array}{ccc} Z & \xrightarrow{q} & X \\ g \downarrow & & \downarrow f \\ B & \xrightarrow{p} & A \end{array}$$

By construction q is also finite and étale. Therefore, we have $q^* \omega_X = \omega_Z$ and $q^* \mathcal{J}(\varphi) = \mathcal{J}(q^* \varphi)$. By the flat base change theorem,

$$p^* R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \simeq R^j g_*(\omega_Z \otimes q^* F \otimes \mathcal{J}(q^* h)).$$

By Theorem D, we obtain the desired vanishing (4.8). \square

Step 6 (Theorems C and E \implies Theorem F). By Theorem C, we have $\mathcal{F}^j := R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) = 0$ for $j > \dim X - \dim f(X)$. We consider the following spectral sequence:

$$E_2^{pq} = H^p(A, \mathcal{F}^q \otimes L) \Rightarrow H^{p+q}(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^* L)$$

for every $L \in \text{Pic}^0(A)$. Note that \mathcal{F}^j is a GV-sheaf for every j and that $\mathcal{F}^j = 0$ for $j > \dim X - \dim f(X)$. Then we obtain

$$\text{codim}_{\text{Pic}^0(A)} \{L \in \text{Pic}^0(A) \mid H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^* L) \neq 0\} \geq i - (\dim X - \dim f(X))$$

for every $i \geq 0$.

We completed the proof of Proposition 1.9. \square

We prove Corollary 1.7 as an application of Theorem D.

Proof of Corollary 1.7 (Theorem D \implies Corollary 1.7). By Theorem D, we have

$$H^p(Y, R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m-p}) = 0$$

for every $p \geq 1$, $i \geq 0$, and $m \geq \dim Y + 1$. Thus the Castelnuovo–Mumford regularity (see [L1, Section 1.8]) implies that $R^i f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}$ is globally generated for every $i \geq 0$ and $m \geq \dim Y + 1$. \square

We close this section with a proof of Theorem 1.4 based on Theorem A for the reader's convenience. Of course, the usual proof of Theorem 1.4 is much easier than the proof of Theorem A given in Section 5.

Proof of Theorem 1.4 (Theorem A \implies Theorem 1.4). Let A be an ample line bundle on V . Then there exists a sufficiently large positive integer m such that $A^{\otimes m}$ is very ample and that $H^i(V, \omega_V \otimes L \otimes \mathcal{J}(h_L) \otimes A^{\otimes m}) = 0$ for every $i > 0$ by the Serre vanishing theorem. We can take a smooth hermitian metric h_A on A such that $\sqrt{-1}\Theta_{h_A}(A)$ is a smooth positive $(1, 1)$ -form on V . Therefore, we have $\sqrt{-1}\Theta_{h_A^m}(A^{\otimes m}) \geq 0$. By the condition $\sqrt{-1}\Theta_{h_L}(L) \geq \varepsilon\omega$, we see that $\sqrt{-1}(\Theta_{h_L}(L) - t\Theta_{h_A^m}(A^{\otimes m})) \geq 0$ for some $0 < t \ll 1$. We take a nonzero global section s of $A^{\otimes m}$. By Theorem A, we see that

$$\times s : H^i(V, \omega_V \otimes L \otimes \mathcal{J}(h_L)) \rightarrow H^i(V, \omega_V \otimes L \otimes \mathcal{J}(h_L) \otimes A^{\otimes m})$$

is injective for every i . Thus we obtain that $H^i(V, \omega_V \otimes L \otimes \mathcal{J}(h_L)) = 0$ for every $i > 0$. \square

Anyway, we see that Theorem A is very powerful and has many applications although the formulation of Theorem A may look a little bit artificial.

5. PROOF OF THEOREM A

In this section, we will give a proof of Theorem A. Precisely speaking, we will prove:

Theorem 5.1 (Theorem A). *Let F (resp. M) be a line bundle on a compact Kähler manifold X with a singular hermitian metric h (resp. a smooth hermitian metric h_M) satisfying*

$$\sqrt{-1}\Theta_{h_M}(M) \geq 0 \text{ and } \sqrt{-1}\Theta_h(F) - b\sqrt{-1}\Theta_{h_M}(M) \geq 0 \text{ for some } b > 0.$$

Then, for a (nonzero) section $s \in H^0(X, M)$, the multiplication map induced by $\otimes s$

$$H^q(X, \omega_X \otimes F \otimes \mathcal{J}(h)) \xrightarrow{\otimes s} H^q(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes M)$$

is injective for every q . Here ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h .

Proof of Theorem 5.1 (Theorem A). In the proof we use the notation in Theorem 5.1. We may assume $q > 0$ since the case $q = 0$ is obvious. The proof can be divided into four steps.

Step 1. Throughout the proof, we fix a Kähler form ω on X . For a given singular hermitian metric h on F , by applying [DPS, Theorem 2.3] to the weight of h , we obtain a family of singular hermitian metrics $\{h_\varepsilon\}_{1 \gg \varepsilon > 0}$ on F with the following properties:

- (a) h_ε is smooth on $Y_\varepsilon := X \setminus Z_\varepsilon$, where Z_ε is a proper closed subvariety on X .
- (b) $h_{\varepsilon'} \leq h_{\varepsilon''} \leq h$ holds on X when $\varepsilon' > \varepsilon'' > 0$.
- (c) $\mathcal{J}(h) = \mathcal{J}(h_\varepsilon)$ on X .
- (d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq b\sqrt{-1}\Theta_{h_M}(M) - \varepsilon\omega$ on X .

Here property (d) is derived from the assumption $\sqrt{-1}\Theta_h(F) \geq b\sqrt{-1}\Theta_{h_M}(M)$.

The main difficulty of the proof is that Z_ε may essentially depend on ε , compared to [MaS4] in which we have already studied the situation where Z_ε is independent of ε . To overcome this difficulty, we consider suitable complete Kähler forms $\{\omega_{\varepsilon,\delta}\}_{\delta>0}$ on Y_ε such that $\omega_{\varepsilon,\delta}$ converges to the fixed ω as δ tends to zero. To construct such complete Kähler forms, we first take a complete Kähler form ω_ε on Y_ε with the following properties:

- ω_ε is a complete Kähler form on Y_ε .
- $\omega_\varepsilon \geq \omega$ on Y_ε .
- $\omega_\varepsilon = \sqrt{-1}\partial\bar{\partial}\Psi_\varepsilon$ for some bounded function Ψ_ε on a neighborhood of every $p \in X$.

See [F4, Section 3] for the construction of ω_ε . For the Kähler form $\omega_{\varepsilon,\delta}$ on Y_ε defined to be

$$\omega_{\varepsilon,\delta} := \omega + \delta\omega_\varepsilon \text{ for } \varepsilon \text{ and } \delta \text{ with } 0 < \delta \ll \varepsilon,$$

it is easy to see the following properties:

- (A) $\omega_{\varepsilon,\delta}$ is a complete Kähler form on $Y_\varepsilon = X \setminus Z_\varepsilon$ for every $\delta > 0$.
- (B) $\omega_{\varepsilon,\delta} \geq \omega$ on Y_ε for every $\delta > 0$.
- (C) $\Psi + \delta\Psi_\varepsilon$ is a bounded local potential function of $\omega_{\varepsilon,\delta}$ and converges to Ψ as $\delta \rightarrow 0$.

Here Ψ is a local potential function of ω . The first property enables us to use the theory of harmonic integrals on the noncompact Y_ε , and the third property enables us to construct the De Rham–Weil isomorphism from the $\bar{\partial}$ -cohomology on Y_ε to the Čech cohomology on X .

Remark 5.2. Strictly speaking, by [DPS, Theorem 2.3], we obtain a countable family $\{h_{\varepsilon_k}\}_{k=1}^\infty$ of singular hermitian metrics satisfying the above properties and $\varepsilon_k \rightarrow 0$. In the proof of Theorem 5.1, we actually consider only a countable sequence $\{\varepsilon_k\}_{k=1}^\infty$ (resp. $\{\delta_\ell\}_{\ell=1}^\infty$) converging to zero since we need to apply Cantor’s diagonal argument, but we often use the notation ε (resp. δ) for simplicity.

For the proof, it is sufficient to show that an arbitrary cohomology class $\eta \in H^q(X, \omega_X \otimes F \otimes \mathcal{J}(h))$ satisfying $s\eta = 0 \in H^q(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes M)$ is actually zero. We represent the cohomology class $\eta \in H^q(X, \omega_X \otimes F \otimes \mathcal{J}(h))$ by a $\bar{\partial}$ -closed F -valued (n, q) -form u with $\|u\|_{h,\omega} < \infty$ by using the standard De Rham–Weil isomorphism

$$H^q(X, \omega_X \otimes F \otimes \mathcal{J}(h)) \cong \frac{\text{Ker } \bar{\partial} : L_{(2)}^{n,q}(F)_{h,\omega} \rightarrow L_{(2)}^{n,q+1}(F)_{h,\omega}}{\text{Im } \bar{\partial} : L_{(2)}^{n,q-1}(F)_{h,\omega} \rightarrow L_{(2)}^{n,q}(F)_{h,\omega}}.$$

Here $\bar{\partial}$ is the densely defined closed operator defined by the usual $\bar{\partial}$ -operator and $L_{(2)}^{n,q}(F)_{h,\omega}$ is the L^2 -space of F -valued (n, q) -forms on X with respect to the L^2 -norm $\|\bullet\|_{h,\omega}$ defined by

$$\|\bullet\|_{h,\omega}^2 := \int_X |\bullet|_{h,\omega}^2 dV_\omega,$$

where $dV_\omega := \omega^n/n!$ and $n := \dim X$. Our purpose is to prove that u is $\bar{\partial}$ -exact (namely, $u \in \text{Im } \bar{\partial} \subset L_{(2)}^{n,q}(F)_{h,\omega}$) under the assumption that the cohomology class of su is zero in $H^q(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes M)$.

From now on, we mainly consider the L^2 -space $L_{(2)}^{n,q}(Y_\varepsilon, F)_{h_\varepsilon, \omega_{\varepsilon, \delta}}$ of F -valued (n, q) -forms on Y_ε (not X) with respect to h_ε and $\omega_{\varepsilon, \delta}$ (not h and ω). For simplicity we put

$$L_{(2)}^{n,q}(F)_{\varepsilon, \delta} := L_{(2)}^{n,q}(Y_\varepsilon, F)_{h_\varepsilon, \omega_{\varepsilon, \delta}} \quad \text{and} \quad \|\bullet\|_{\varepsilon, \delta} := \|\bullet\|_{h_\varepsilon, \omega_{\varepsilon, \delta}}.$$

The following inequality plays an important role in the proof.

$$(5.1) \quad \|u\|_{\varepsilon, \delta} \leq \|u\|_{h, \omega_{\varepsilon, \delta}} \leq \|u\|_{h, \omega} < \infty.$$

In particular, the norm $\|u\|_{\varepsilon, \delta}$ is uniformly bounded since the right hand side is independent of ε, δ . The first inequality follows from property (b) of h_ε , and the second inequality follows from Lemma 2.4 and property (B) of $\omega_{\varepsilon, \delta}$. Here we used a special characteristic of the canonical bundle ω_X since the second inequality holds only for (n, q) -forms. Strictly speaking, the left hand side should be $\|u|_{Y_\varepsilon}\|_{\varepsilon, \delta}$, but we often omit the symbol of restriction. Now we have the following orthogonal decomposition (for example see [MaS4, Proposition 5.8]).

$$L_{(2)}^{n,q}(F)_{\varepsilon, \delta} = \text{Im } \bar{\partial} \oplus \mathcal{H}_{\varepsilon, \delta}^{n,q}(F) \oplus \text{Im } \bar{\partial}_{\varepsilon, \delta}^*.$$

Here $\bar{\partial}_{\varepsilon, \delta}^*$ is (the maximal extension of) the formal adjoint of the $\bar{\partial}$ -operator and $\mathcal{H}_{\varepsilon, \delta}^{n,q}(F)$ is the set of harmonic F -valued (n, q) -forms on Y_ε , namely

$$\mathcal{H}_{\varepsilon, \delta}^{n,q}(F) := \{w \in L_{(2)}^{n,q}(F)_{\varepsilon, \delta} \mid \bar{\partial}w = 0 \text{ and } \bar{\partial}_{\varepsilon, \delta}^*w = 0\}.$$

Remark 5.3. The formal adjoint coincides with the Hilbert space adjoint since $\omega_{\varepsilon, \delta}$ is complete for $\delta > 0$ (see, for example, [D4, (3.2) Theorem in Chapter VIII]). Strictly speaking, $\bar{\partial}$ also depends on h_ε and $\omega_{\varepsilon, \delta}$ since the domain and range of the closed operator $\bar{\partial}$ depend on them, but we abbreviate $\bar{\partial}_{\varepsilon, \delta}$ to $\bar{\partial}$.

The F -valued (n, q) -form u (representing η) belongs to $L_{(2)}^{n,q}(F)_{\varepsilon, \delta}$ by (5.1), and thus u can be decomposed as follows:

$$(5.2) \quad u = \bar{\partial}w_{\varepsilon, \delta} + u_{\varepsilon, \delta} \quad \text{for some } w_{\varepsilon, \delta} \in \text{Dom } \bar{\partial} \subset L_{(2)}^{n,q-1}(F)_{\varepsilon, \delta} \text{ and } u_{\varepsilon, \delta} \in \mathcal{H}_{\varepsilon, \delta}^{n,q}(F).$$

Note that the orthogonal projection of u to $\text{Im } \bar{\partial}_{\varepsilon, \delta}^*$ must be zero since u is $\bar{\partial}$ -closed.

Step 2. The purpose of this step is to prove Proposition 5.7, which reduces the proof to study the asymptotic behavior of the norm of $su_{\varepsilon, \delta}$. When we consider a suitable limit of $u_{\varepsilon, \delta}$ in the following proposition, we need to carefully choose the L^2 -space since the L^2 -space $L_{(2)}^{n,q}(F)_{\varepsilon, \delta}$ depends on ε, δ . We remark that $\{\varepsilon\}_{\varepsilon>0}$ and $\{\delta\}_{\delta>0}$ denote countable sequences converging to zero (see Remark 5.2). Let $\{\delta_0\}_{\delta_0>0}$ denote another countable sequence converging to zero.

Proposition 5.4. *There exist a subsequence $\{\delta_\nu\}_{\nu=1}^\infty$ of $\{\delta\}_{\delta>0}$ and $\alpha_\varepsilon \in L_{(2)}^{n,q}(F)_{h_\varepsilon, \omega}$ with the following properties:*

- For any $\varepsilon, \delta_0 > 0$, as δ_ν tends to 0,

$$u_{\varepsilon, \delta_\nu} \text{ converges to } \alpha_\varepsilon \text{ with respect to the weak } L^2\text{-topology in } L_{(2)}^{n,q}(F)_{\varepsilon, \delta_0}.$$

- For any $\varepsilon > 0$,

$$\|\alpha_\varepsilon\|_{h_\varepsilon, \omega} \leq \varliminf_{\delta_0 \rightarrow 0} \|\alpha_\varepsilon\|_{\varepsilon, \delta_0} \leq \varliminf_{\delta_\nu \rightarrow 0} \|u_{\varepsilon, \delta_\nu}\|_{\varepsilon, \delta_\nu} \leq \|u\|_{h, \omega}.$$

Remark 5.5. The weak limit α_ε does not depend on δ_0 , and the subsequence $\{\delta_\nu\}_{\nu=1}^\infty$ does not depend on ε, δ_0 .

Proof of Proposition 5.4. For ε and δ_0 , by taking a sufficiently small δ with $0 < \delta < \delta_0$, we have

$$(5.3) \quad \|u_{\varepsilon,\delta}\|_{\varepsilon,\delta_0} \leq \|u_{\varepsilon,\delta}\|_{\varepsilon,\delta} \leq \|u\|_{\varepsilon,\delta} \leq \|u\|_{h,\omega}.$$

The first inequality follows from $\omega_{\varepsilon,\delta} \leq \omega_{\varepsilon,\delta_0}$ and Lemma 2.4, the second inequality follows since $u_{\varepsilon,\delta}$ is the orthogonal projection of u with respect to ε, δ , and the last inequality follows from (5.1). Since the right hand side is independent of δ , the family $\{u_{\varepsilon,\delta}\}_{\delta>0}$ is uniformly bounded in $L_{(2)}^{n,q}(F)_{\varepsilon,\delta_0}$. Therefore, there exists a subsequence $\{\delta_\nu\}_{\nu=1}^\infty$ of $\{\delta\}_{\delta>0}$ such that $u_{\varepsilon,\delta_\nu}$ converges to $\alpha_{\varepsilon,\delta_0}$ with respect to the weak L^2 -topology in $L_{(2)}^{n,q}(F)_{\varepsilon,\delta_0}$. This subsequence $\{\delta_\nu\}_{\nu=1}^\infty$ may depend on ε, δ_0 , but we can choose a subsequence independent of them by applying Cantor's diagonal argument.

Now we show that $\alpha_{\varepsilon,\delta_0}$ does not depend on δ_0 . For arbitrary δ'_0, δ''_0 with $0 < \delta'_0 \leq \delta''_0$, the natural inclusion $L_{(2)}^{n,q}(F)_{\varepsilon,\delta'_0} \rightarrow L_{(2)}^{n,q}(F)_{\varepsilon,\delta''_0}$ is a bounded operator (continuous linear map) by $\|\bullet\|_{\varepsilon,\delta''_0} \leq \|\bullet\|_{\varepsilon,\delta'_0}$, and thus $u_{\varepsilon,\delta_\nu}$ weakly converges to $\alpha_{\varepsilon,\delta'_0}$ in not only $L_{(2)}^{n,q}(F)_{\varepsilon,\delta'_0}$ but also $L_{(2)}^{n,q}(F)_{\varepsilon,\delta''_0}$ by Lemma 2.5. We have $\alpha_{\varepsilon,\delta'_0} = \alpha_{\varepsilon,\delta''_0}$ since the weak limit is unique.

Finally we consider the norm of α_ε . It is easy to see

$$\|\alpha_\varepsilon\|_{\varepsilon,\delta_0} \leq \varliminf_{\delta_\nu \rightarrow 0} \|u_{\varepsilon,\delta_\nu}\|_{\varepsilon,\delta_0} \leq \varliminf_{\delta_\nu \rightarrow 0} \|u_{\varepsilon,\delta_\nu}\|_{\varepsilon,\delta_\nu} \leq \|u\|_{h,\omega}.$$

The first inequality follows since the norm is lower semicontinuous with respect to the weak convergence, the second inequality follows from $\omega_{\varepsilon,\delta_0} \geq \omega_{\varepsilon,\delta_\nu}$, and the last inequality follows from (5.3). Fatou's lemma yields

$$\|\alpha_\varepsilon\|_{h_\varepsilon,\omega}^2 = \int_{Y_\varepsilon} |\alpha_\varepsilon|_{h_\varepsilon,\omega}^2 dV_\omega \leq \varliminf_{\delta_0 \rightarrow 0} \int_{Y_\varepsilon} |\alpha_\varepsilon|_{h_\varepsilon,\omega_{\varepsilon,\delta_0}}^2 dV_{\omega_{\varepsilon,\delta_0}} = \varliminf_{\delta_0 \rightarrow 0} \|\alpha_\varepsilon\|_{\varepsilon,\delta_0}^2.$$

These inequalities lead to the desired estimate in the proposition. \square

For simplicity, we use the same notation $\{u_{\varepsilon,\delta}\}_{\delta>0}$ for the subsequence $\{u_{\varepsilon,\delta_\nu}\}_{\nu=1}^\infty$ in Proposition 5.4. We fix $\varepsilon_0 > 0$ and consider the weak limit of α_ε in the fixed L^2 -space $L_{(2)}^{n,q}(F)_{h_{\varepsilon_0},\omega}$. For a sufficiently small $\varepsilon > 0$, we have

$$\|\alpha_\varepsilon\|_{h_{\varepsilon_0},\omega} \leq \|\alpha_\varepsilon\|_{h_\varepsilon,\omega} \leq \|u\|_{h,\omega}$$

by property (b) and Proposition 5.4. By taking a subsequence of $\{\alpha_\varepsilon\}_{\varepsilon>0}$, we may assume that α_ε weakly converges to some α in $L_{(2)}^{n,q}(F)_{h_{\varepsilon_0},\omega}$.

Proposition 5.6. *If the weak limit α is zero in $L_{(2)}^{n,q}(F)_{h_{\varepsilon_0},\omega}$, then the cohomology class η is zero in $H^q(X, \omega_X \otimes F \otimes \mathcal{J}(h))$.*

Proof of Proposition 5.6. For every δ with $0 < \delta \leq \delta_0$, we can easily check

$$u - u_{\varepsilon,\delta} \in \text{Im } \bar{\partial} \text{ in } L_{(2)}^{n,q}(F)_{\varepsilon,\delta} \subset \text{Im } \bar{\partial} \text{ in } L_{(2)}^{n,q}(F)_{\varepsilon,\delta_0}.$$

from the construction of $u_{\varepsilon,\delta}$. As δ tends to zero, we obtain

$$u - \alpha_\varepsilon \in \text{Im } \bar{\partial} \text{ in } L_{(2)}^{n,q}(F)_{\varepsilon,\delta_0}$$

by Lemma 2.6 and Proposition 5.4. We remark that $\text{Im } \bar{\partial}$ is a closed subspace (see [MaS4, Proposition 5.8]). On the other hand, we have the following commutative diagram:

$$\begin{array}{ccccc}
 \text{Ker } \bar{\partial} \text{ in } L_{(2)}^{n,q}(F)_{\varepsilon,\delta_0} & \xrightarrow{q_1} & \frac{\text{Ker } \bar{\partial}}{\text{Im } \bar{\partial}} \text{ of } L_{(2)}^{n,q}(F)_{\varepsilon,\delta_0} & \xrightarrow[\cong]{f_1} & \check{H}^q(X, \omega_X \otimes F \otimes \mathcal{J}(h)) \\
 \uparrow j_1 & & & & \uparrow \cong f_2 \\
 \text{Ker } \bar{\partial} \text{ in } L_{(2)}^{n,q}(F)_{h_\varepsilon,\omega} & \xrightarrow{j_2} & \text{Ker } \bar{\partial} \text{ in } L_{(2)}^{n,q}(F)_{h_{\varepsilon_0},\omega} & \xrightarrow{q_2} & \frac{\text{Ker } \bar{\partial}}{\text{Im } \bar{\partial}} \text{ of } L_{(2)}^{n,q}(F)_{h_{\varepsilon_0},\omega}
 \end{array}$$

Here j_1, j_2 are the natural inclusions, q_1, q_2 are the natural quotient maps, and f_1, f_2 are the De Rham–Weil isomorphisms (see [MaS4, Proposition 5.5] for the construction). Strictly speaking, f_1 is an isomorphism to $\check{H}^q(X, \omega_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$, but which coincides with $\check{H}^q(X, \omega_X \otimes F \otimes \mathcal{J}(h))$ by property (c). To check that j_2 is well-defined, we have to see that $\bar{\partial}w = 0$ on Y_{ε_0} if $\bar{\partial}w = 0$ on Y_ε . By the L^2 -integrability and [D4, (7.3) Lemma, Chapter VIII], the equality $\bar{\partial}w = 0$ can be extended from Y_ε to X (in particular Y_{ε_0}). The key point here is the L^2 -integrability with respect to ω (not $\omega_{\varepsilon,\delta}$).

Since $j_2(u - \alpha_\varepsilon)$ weakly converges to $j_2(u - \alpha)$ and the $\bar{\partial}$ -cohomology is finite dimensional, we obtain

$$\lim_{\varepsilon \rightarrow 0} q_2(u - \alpha_\varepsilon) = q_2(u - \alpha) = q_2(u)$$

by Lemma 2.5 and the assumption $\alpha = 0$. On the other hand, we know that $q_1(u - \alpha_\varepsilon) = 0$ by the first half argument. Hence we have $q_2(u) = 0$, that is, $u \in \text{Im } \bar{\partial} \subset L_{(2)}^{n,q}(F)_{h_{\varepsilon_0},\omega}$. From $q_2(u) = 0$, we can prove the conclusion, that is, $u \in \text{Im } \bar{\partial} \subset L_{(2)}^{n,q}(F)_{h,\omega}$. Indeed, we can obtain $q_3(u) = 0$ (which leads to the conclusion) by the following commutative diagram:

$$\begin{array}{ccccc}
 \text{Ker } \bar{\partial} \text{ in } L_{(2)}^{n,q}(F)_{h_{\varepsilon_0},\omega} & \xrightarrow{q_2} & \frac{\text{Ker } \bar{\partial}}{\text{Im } \bar{\partial}} \text{ of } L_{(2)}^{n,q}(F)_{h_{\varepsilon_0},\omega} & \xrightarrow[\cong]{f_2} & \check{H}^q(X, \omega_X \otimes F \otimes \mathcal{J}(h_{\varepsilon_0})) \\
 \uparrow & & & & \parallel \\
 \text{Ker } \bar{\partial} \text{ in } L_{(2)}^{n,q}(F)_{h,\omega} & \xrightarrow{q_3} & \frac{\text{Ker } \bar{\partial}}{\text{Im } \bar{\partial}} \text{ of } L_{(2)}^{n,q}(F)_{h,\omega} & \xrightarrow[\cong]{f_3} & \check{H}^q(X, \omega_X \otimes F \otimes \mathcal{J}(h)).
 \end{array}$$

□

At the end of this step, we prove Proposition 5.7. For simplicity, we write the norm with respect to $h_\varepsilon h_M$ and $\omega_{\varepsilon,\delta}$ as

$$\|\bullet\|_{\varepsilon,\delta} := \|\bullet\|_{h_\varepsilon h_M, \omega_{\varepsilon,\delta}} \text{ for an } F \otimes M\text{-valued form } \bullet.$$

Proposition 5.7. *If we have*

$$\lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \|su_{\varepsilon,\delta}\|_{\varepsilon,\delta} = 0,$$

then the weak limit α is zero. In particular, the cohomology class η is zero by Proposition 5.6.

Proof of Proposition 5.7. In the proof we compare the norm of $u_{\varepsilon,\delta}$ with the norm of $su_{\varepsilon,\delta}$. For this purpose, we define $Y_{\varepsilon_0}^k$ to be

$$Y_{\varepsilon_0}^k := \{y \in Y_{\varepsilon_0} \mid |s|_{h_M} > 1/k \text{ at } y\}$$

for $k \gg 0$. Note the subset $Y_{\varepsilon_0}^k$ is an open set in Y_{ε_0} . It follows that the restriction $\alpha_\varepsilon|_{Y_{\varepsilon_0}^k}$ also weakly converges to $\alpha|_{Y_{\varepsilon_0}^k}$ in $L_{(2)}^{n,q}(Y_{\varepsilon_0}^k, F)_{h_{\varepsilon_0}, \omega}$ since the restriction map $L_{(2)}^{n,q}(F)_{h_{\varepsilon_0}, \omega} \rightarrow L_{(2)}^{n,q}(Y_{\varepsilon_0}^k, F)_{h_{\varepsilon_0}, \omega}$ is a bounded operator and α_ε weakly converges to α in $L_{(2)}^{n,q}(F)_{h_{\varepsilon_0}, \omega}$. Since the norm is lower semicontinuous with respect to the weak convergence, we obtain the estimate for the L^2 -norm on $Y_{\varepsilon_0}^k$

$$\|\alpha\|_{Y_{\varepsilon_0}^k, h_{\varepsilon_0}, \omega} \leq \varliminf_{\varepsilon \rightarrow 0} \|\alpha_\varepsilon\|_{Y_{\varepsilon_0}^k, h_{\varepsilon_0}, \omega} \leq \varliminf_{\varepsilon \rightarrow 0} \|\alpha_\varepsilon\|_{Y_{\varepsilon_0}^k, h_{\varepsilon}, \omega}$$

by property (b). By the same argument, the restriction $u_{\varepsilon, \delta}|_{Y_{\varepsilon_0}^k}$ weakly converges to $\alpha_\varepsilon|_{Y_{\varepsilon_0}^k}$ in $L_{(2)}^{n,q}(Y_{\varepsilon_0}^k, F)_{\varepsilon, \delta_0}$, and thus we obtain

$$\|\alpha_\varepsilon\|_{Y_{\varepsilon_0}^k, \varepsilon, \delta_0} \leq \varliminf_{\delta \rightarrow 0} \|u_{\varepsilon, \delta}\|_{Y_{\varepsilon_0}^k, \varepsilon, \delta_0} \leq \varliminf_{\delta \rightarrow 0} \|u_{\varepsilon, \delta}\|_{Y_{\varepsilon_0}^k, \varepsilon, \delta}$$

by Lemma 2.4. As δ_0 tends to zero in the above inequality, we have

$$\|\alpha_\varepsilon\|_{Y_{\varepsilon_0}^k, h_{\varepsilon}, \omega} \leq \varliminf_{\delta_0 \rightarrow 0} \|\alpha_\varepsilon\|_{Y_{\varepsilon_0}^k, \varepsilon, \delta_0} \leq \varliminf_{\delta \rightarrow 0} \|u_{\varepsilon, \delta}\|_{Y_{\varepsilon_0}^k, \varepsilon, \delta}.$$

by Fatou's lemma (see the argument in Proposition 5.4). These inequalities yield

$$\|\alpha\|_{Y_{\varepsilon_0}^k, h_{\varepsilon_0}, \omega} \leq \varliminf_{\varepsilon \rightarrow 0} \varliminf_{\delta \rightarrow 0} \|u_{\varepsilon, \delta}\|_{Y_{\varepsilon_0}^k, \varepsilon, \delta}.$$

On the other hand, it follows

$$\|u_{\varepsilon, \delta}\|_{Y_{\varepsilon_0}^k, \varepsilon, \delta} \leq k \|su_{\varepsilon, \delta}\|_{Y_{\varepsilon_0}^k, \varepsilon, \delta} \leq k \|su_{\varepsilon, \delta}\|_{\varepsilon, \delta}$$

from $1/k < |s|_{h_M}$ on $Y_{\varepsilon_0}^k$. This implies that $\alpha = 0$ on $Y_{\varepsilon_0}^k$ for an arbitrary $k \gg 0$. From $\bigcup_{k \gg 0} Y_{\varepsilon_0}^k = Y_{\varepsilon_0} \setminus \{s = 0\}$, we obtain the conclusion. \square

Step 3. The purpose of this step is to prove the following proposition:

Proposition 5.8.

$$\lim_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \|\bar{\partial}_{\varepsilon, \delta}^* su_{\varepsilon, \delta}\|_{\varepsilon, \delta} = 0.$$

Proof of Proposition 5.8. In the proof, we will often use (5.3). By applying Bochner–Kodaira–Nakano's identity and the density lemma to $u_{\varepsilon, \delta}$ and $su_{\varepsilon, \delta}$ (see [MaS1, Proposition 2.8]), we obtain

$$(5.4) \quad 0 = \langle \sqrt{-1} \Theta_{h_\varepsilon}(F) \Lambda_{\omega_{\varepsilon, \delta}} u_{\varepsilon, \delta}, u_{\varepsilon, \delta} \rangle_{\varepsilon, \delta} + \|D_{\varepsilon, \delta}'^* u_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2,$$

$$(5.5) \quad \|\bar{\partial}_{\varepsilon, \delta}^* su_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 = \langle \sqrt{-1} \Theta_{h_M}(F \otimes M) \Lambda_{\omega_{\varepsilon, \delta}} su_{\varepsilon, \delta}, su_{\varepsilon, \delta} \rangle_{\varepsilon, \delta} + \|D_{\varepsilon, \delta}'^* su_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2,$$

where $D_{\varepsilon, \delta}'^*$ is the adjoint operator of the $(1, 0)$ -part of the Chern connection D_{h_ε} . Here we used the fact that $u_{\varepsilon, \delta}$ is harmonic and $\bar{\partial}(su_{\varepsilon, \delta}) = s\bar{\partial}u_{\varepsilon, \delta} = 0$. Now we have

$$\sqrt{-1} \Theta_{h_\varepsilon}(F) \geq b\sqrt{-1} \Theta_{h_M}(M) - \varepsilon\omega \geq -\varepsilon\omega \geq -\varepsilon\omega_{\varepsilon, \delta}$$

by property (d) and property (B). Hence the integrand $g_{\varepsilon, \delta}$ of the first term of (5.4) satisfies

$$(5.6) \quad -\varepsilon q|u_{\varepsilon, \delta}|_{\varepsilon, \delta}^2 \leq g_{\varepsilon, \delta} := \langle \sqrt{-1} \Theta_{h_\varepsilon}(F) \Lambda_{\omega_{\varepsilon, \delta}} u_{\varepsilon, \delta}, u_{\varepsilon, \delta} \rangle_{\varepsilon, \delta}.$$

For the precise argument, see [MaS4, Step 2 in the proof of Theorem 3.1]. Then, by (5.4), we can easily see

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \left(\int_{\{g_{\varepsilon, \delta} \geq 0\}} g_{\varepsilon, \delta} dV_{\omega_{\varepsilon, \delta}} + \|D'_{\varepsilon, \delta} u_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 \right) &= \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \left(- \int_{\{g_{\varepsilon, \delta} \leq 0\}} g_{\varepsilon, \delta} dV_{\omega_{\varepsilon, \delta}} \right) \\ &\leq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \left(\varepsilon q \int_{\{g_{\varepsilon, \delta} \leq 0\}} |u_{\varepsilon, \delta}|_{\varepsilon, \delta}^2 dV_{\omega_{\varepsilon, \delta}} \right) \\ &\leq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \left(\varepsilon q \|u_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 \right) = 0. \end{aligned}$$

Here we used (5.3) in the last equality.

On the other hand, by $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq b\sqrt{-1}\Theta_{h_M}(M) - \varepsilon\omega_{\varepsilon, \delta}$, we have

$$\begin{aligned} &\langle \sqrt{-1}\Theta_{h_\varepsilon h_M}(F \otimes M)\Lambda_{\omega_{\varepsilon, \delta}} su_{\varepsilon, \delta}, su_{\varepsilon, \delta} \rangle_{\varepsilon, \delta} \\ &\leq \left(1 + \frac{1}{b}\right) \int_{Y_\varepsilon} |s|_{h_M}^2 g_{\varepsilon, \delta} dV_{\omega_{\varepsilon, \delta}} + \frac{\varepsilon q}{b} \int_{Y_\varepsilon} |s|_{h_M}^2 |u_{\varepsilon, \delta}|_{\varepsilon, \delta}^2 dV_{\omega_{\varepsilon, \delta}} \\ &\leq \left(1 + \frac{1}{b}\right) \sup_X |s|_{h_M}^2 \int_{\{g_{\varepsilon, \delta} \geq 0\}} g_{\varepsilon, \delta} dV_{\omega_{\varepsilon, \delta}} + \frac{\varepsilon q}{b} \sup_X |s|_{h_M}^2 \|u_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2. \end{aligned}$$

Furthermore, since $D'_{\varepsilon, \delta}$ can be expressed as $D'_{\varepsilon, \delta} = - * \bar{\partial}^*$ by the Hodge star operator $*$ with respect to $\omega_{\varepsilon, \delta}$, we have

$$\|D'_{\varepsilon, \delta} su_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 = \|s D'_{\varepsilon, \delta} u_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 \leq \sup_X |s|_{h_M}^2 \|D'_{\varepsilon, \delta} u_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2.$$

The right hand side of (5.5) can be shown to converge to zero by the first half argument and these inequalities. \square

Step 4. In this step, we construct solutions $v_{\varepsilon, \delta}$ of the $\bar{\partial}$ -equation $\bar{\partial} v_{\varepsilon, \delta} = su_{\varepsilon, \delta}$ with suitable L^2 -norm, and we finish the proof of Theorem 5.1. The proof of the following proposition is a slight variant of that of [MaS4, Theorem 5.9].

Proposition 5.9. *There exist F -valued $(n, q-1)$ -forms $w_{\varepsilon, \delta}$ on Y_ε with the following properties:*

- $\bar{\partial} w_{\varepsilon, \delta} = u - u_{\varepsilon, \delta}$.
- $\lim_{\delta \rightarrow 0} \|w_{\varepsilon, \delta}\|_{\varepsilon, \delta}$ can be bounded by a constant independent of ε .

Before we begin to prove Proposition 5.9, we recall the content in [MaS4, Section 5] with our notation. For a finite open cover $\mathcal{U} := \{B_i\}_{i \in I}$ of X by sufficiently small Stein open sets B_i , we can construct

$$f_{\varepsilon, \delta} : \text{Ker } \bar{\partial} \text{ in } L_{(2)}^{n, q}(F)_{\varepsilon, \delta} \longrightarrow \text{Ker } \mu \text{ in } C^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$$

such that $f_{\varepsilon, \delta}$ induces the De Rham–Weil isomorphism

$$(5.7) \quad \overline{f_{\varepsilon, \delta}} : \frac{\text{Ker } \bar{\partial}}{\text{Im } \bar{\partial}} \text{ of } L_{(2)}^{n, q}(F)_{\varepsilon, \delta} \xrightarrow{\cong} \frac{\text{Ker } \mu}{\text{Im } \mu} \text{ of } C^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h_\varepsilon)).$$

Here $C^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$ is the space of q -cochains calculated by \mathcal{U} and μ is the coboundary operator. We remark that $C^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$ is a Fréchet space with respect to the seminorm $p_{K_{i_0 \dots i_q}}(\bullet)$ defined to be

$$p_{K_{i_0 \dots i_q}}(\{\beta_{i_0 \dots i_q}\})^2 := \int_{K_{i_0 \dots i_q}} |\beta_{i_0 \dots i_q}|_{h_\varepsilon, \omega}^2 dV_\omega$$

for a relatively compact set $K_{i_0 \dots i_q} \Subset B_{i_0 \dots i_q} := B_{i_0} \cap \dots \cap B_{i_q}$ (see [MaS4, Theorem 5.3]). The construction of $f_{\varepsilon, \delta}$ is essentially the same as in the proof of [MaS4, Proposition 5.5]. The only difference is that we use Lemma 5.12 instead of [MaS4, Lemma 5.4] when we locally solve the $\bar{\partial}$ -equation to construct $f_{\varepsilon, \delta}$. Lemma 5.12 will be given at the end of this step. We prove Proposition 5.9 by replacing some constants appearing in the proof of [MaS4, Theorem 5.9] with $C_{\varepsilon, \delta}$ appearing in Lemma 5.12.

Proof of Proposition 5.9. We put $U_{\varepsilon, \delta} := u - u_{\varepsilon, \delta} \in \text{Im } \bar{\partial} \subset L_{(2)}^{n, q}(F)_{\varepsilon, \delta}$. Then there exist the F -valued $(n, q - k - 1)$ -forms $\beta_{i_0 \dots i_k}^{\varepsilon, \delta}$ on $B_{i_0 \dots i_k} \setminus Z_\varepsilon$ satisfying

$$(*) \quad \begin{cases} \bar{\partial} \beta_{i_0}^{\varepsilon, \delta} &= U_{\varepsilon, \delta}|_{B_{i_0} \setminus Z_\varepsilon}, \\ \bar{\partial} \{\beta_{i_0 i_1}^{\varepsilon, \delta}\} &= \mu \{\beta_{i_0}^{\varepsilon, \delta}\}, \\ \bar{\partial} \{\beta_{i_0 i_1 i_2}^{\varepsilon, \delta}\} &= \mu \{\beta_{i_0 i_1}^{\varepsilon, \delta}\}, \\ &\vdots \\ \bar{\partial} \{\beta_{i_0 \dots i_{q-1}}^{\varepsilon, \delta}\} &= \mu \{\beta_{i_0 \dots i_{q-2}}^{\varepsilon, \delta}\}, \\ f_{\varepsilon, \delta}(U_{\varepsilon, \delta}) &= \mu \{\beta_{i_0 \dots i_{q-1}}^{\varepsilon, \delta}\}. \end{cases}$$

Here $\beta_{i_0 \dots i_k}^{\varepsilon, \delta}$ is the solution of the above equation whose norm is minimum among all the solutions (see the construction of $f_{\varepsilon, \delta}$ in [MaS4, Proposition 5.5]). For example, $\beta_{i_0}^{\varepsilon, \delta}$ is the solution of $\bar{\partial} \beta_{i_0}^{\varepsilon, \delta} = U_{\varepsilon, \delta}$ on $B_{i_0} \setminus Z_\varepsilon$ whose norm $\|\beta_{i_0}^{\varepsilon, \delta}\|_{\varepsilon, \delta}$ is minimum among all the solutions. In particular $\|\beta_{i_0}^{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 \leq C_{\varepsilon, \delta} \|U_{\varepsilon, \delta}\|_{B_{i_0, \varepsilon, \delta}}^2 \leq C_{\varepsilon, \delta} \|U_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2$ holds for some constant $C_{\varepsilon, \delta}$ by Lemma 5.12, where $C_{\varepsilon, \delta}$ is a constant such that $\lim_{\delta \rightarrow 0} C_{\varepsilon, \delta}$ (is finite and) is independent of ε . Similarly, $\beta_{i_0 i_1}^{\varepsilon, \delta}$ is the solution of $\bar{\partial} \beta_{i_0 i_1}^{\varepsilon, \delta} = (\beta_{i_1}^{\varepsilon, \delta} - \beta_{i_0}^{\varepsilon, \delta})$ on $B_{i_0 i_1} \setminus Z_\varepsilon$ and the norm

$$\|\beta_{i_0 i_1}^{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 := \int_{B_{i_0 i_1} \setminus Z_\varepsilon} |\beta_{i_0 i_1}^{\varepsilon, \delta}|_{\varepsilon, \delta}^2 dV_{\varepsilon, \delta}$$

is minimum among all the solutions. In particular, $\|\beta_{i_0 i_1}^{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 \leq D_{\varepsilon, \delta} \|(\beta_{i_1}^{\varepsilon, \delta} - \beta_{i_0}^{\varepsilon, \delta})\|_{\varepsilon, \delta}^2$ holds for some constant $D_{\varepsilon, \delta}$ by Lemma 5.12. Of course $D_{\varepsilon, \delta}$ is a constant such that $\lim_{\delta \rightarrow 0} D_{\varepsilon, \delta}$ (is finite and) is independent of ε . Hence we have

$$\|\beta_{i_0 i_1}^{\varepsilon, \delta}\|_{\varepsilon, \delta} \leq D_{\varepsilon, \delta}^{1/2} \|(\beta_{i_1}^{\varepsilon, \delta} - \beta_{i_0}^{\varepsilon, \delta})\|_{\varepsilon, \delta} \leq 2C_{\varepsilon, \delta}^{1/2} D_{\varepsilon, \delta}^{1/2} \|U_{\varepsilon, \delta}\|_{\varepsilon, \delta} \leq 4C_{\varepsilon, \delta}^{1/2} D_{\varepsilon, \delta}^{1/2} \|u\|_{h, \omega}$$

by (5.3). From now on, the notation $C_{\varepsilon, \delta}$ denotes a (possibly different) constant such that $\lim_{\delta \rightarrow 0} C_{\varepsilon, \delta}$ can be bounded by a constant independent of ε . By repeating this process, we have

$$\|\beta_{i_0 \dots i_k}^{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 \leq C_{\varepsilon, \delta} \|u\|_{h, \omega}^2.$$

Moreover, by property (c), we have

$$\alpha_{\varepsilon, \delta} := f_{\varepsilon, \delta}(U_{\varepsilon, \delta}) = \mu \{\beta_{i_0 \dots i_{q-1}}^{\varepsilon, \delta}\} \in C^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h_\varepsilon)) = C^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h)).$$

Claim. *There exist subsequences $\{\varepsilon_k\}_{k=1}^\infty$ and $\{\delta_\ell\}_{\ell=1}^\infty$ with the following properties:*

- $\alpha_{\varepsilon_k, \delta_\ell} \rightarrow \alpha_{\varepsilon_k, 0}$ in $C^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h))$ as $\delta_\ell \rightarrow 0$.
- $\alpha_{\varepsilon_k, 0} \rightarrow \alpha_{0, 0}$ in $C^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h))$ as $\varepsilon_k \rightarrow 0$.

Moreover, the limit $\alpha_{0, 0}$ belongs to $B^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h)) := \text{Im } \mu$.

Proof of Claim. By construction, the norm $\|a_{\varepsilon,\delta}\|_{B_{i_0\dots i_q},\varepsilon,\delta}$ of a component $a_{\varepsilon,\delta} := \alpha_{i_0\dots i_q}^{\varepsilon,\delta}$ of $\alpha_{\varepsilon,\delta} = \{\alpha_{i_0\dots i_q}^{\varepsilon,\delta}\}$ can be bounded by a constant $C_{\varepsilon,\delta}$. Note that $a_{\varepsilon,\delta}$ can be regarded as a holomorphic function on $B_{i_0\dots i_q} \setminus Z_\varepsilon$ with bounded L^2 -norm since it is a $\bar{\partial}$ -closed F -valued $(n,0)$ -form such that $\|a_{\varepsilon,\delta}\|_{B_{i_0\dots i_q},\varepsilon,\delta} < \infty$ (see Lemma 2.4). Hence $a_{\varepsilon,\delta}$ can be extended from $B_{i_0\dots i_q} \setminus Z_\varepsilon$ to $B_{i_0\dots i_q}$ by the Riemann extension theorem. The sup-norm $\sup_K |a_{\varepsilon,\delta}|$ is uniformly bounded with respect to δ for every $K \Subset B_{i_0\dots i_q}$ since the local sup-norm of holomorphic functions can be bounded by the L^2 -norm. By Montel's theorem, we can take a subsequence $\{\delta_\ell\}_{\ell=1}^\infty$ with the first property. This subsequence may depend on ε , but we can take $\{\delta_\ell\}_{\ell=1}^\infty$ independent of (countably many) ε . Then the norm of the limit $a_{\varepsilon,0}$ is uniformly bounded with respect to ε since $\overline{\lim}_{\delta \rightarrow 0} C_{\varepsilon,\delta}$ can be bounded by a constant independent of ε (see Lemma 5.12). Therefore, by applying Montel's theorem again, we can take a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ with the second property. We remark that the convergence with respect to the sup-norm implies the convergence with respect to the local L^2 -norm $p_K(\bullet)$ (see [MaS4, Lemma 5.2]).

It is easy to check the latter conclusion. Indeed, it follows that $\alpha_{\varepsilon,\delta} = f_{\varepsilon,\delta}(U_{\varepsilon,\delta}) \in \text{Im } \mu$ since $U_{\varepsilon,\delta} \in \text{Im } \bar{\partial} \subset L_{(2)}^{n,q}(F)_{\varepsilon,\delta}$ and $f_{\varepsilon,\delta}$ induces the De Rham–Weil isomorphism. By [MaS4, Lemma 5.7], the subspace $\text{Im } \mu$ is closed. Therefore we obtain the latter conclusion. \square

Now, we construct solutions $\gamma_{\varepsilon,\delta}$ of the equation $\mu\gamma_{\varepsilon,\delta} = \alpha_{\varepsilon,\delta}$ with suitable L^2 -norm. For simplicity, we continue to use the same notation for the subsequences in Claim. By the latter conclusion of the claim, there exists $\gamma \in C^{q-1}(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h))$ such that $\mu\gamma = \alpha_{0,0}$. The coboundary operator

$$\mu : C^{q-1}(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h)) \rightarrow B^q(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h)) = \text{Im } \mu$$

is a surjective bounded operator between Fréchet spaces (see [MaS4, Lemma 5.7]), and thus it is an open map by the open mapping theorem. Therefore $\mu(\Delta_K)$ is an open neighborhood of the limit $\alpha_{0,0}$ in $\text{Im } \mu$, where Δ_K is the open bounded neighborhood of γ in $C^{q-1}(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h))$ defined to be

$$\Delta_K := \{\beta \in C^{q-1}(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h)) \mid p_{K_{i_0\dots i_{q-1}}}(\beta - \gamma) < 1\}$$

for a family $K := \{K_{i_0\dots i_{q-1}}\}$ of relatively compact sets $K_{i_0\dots i_{q-1}} \Subset B_{i_0\dots i_{q-1}}$. We have $\alpha_{\varepsilon,\delta} \in \mu(\Delta_K)$ for sufficiently small $\varepsilon, \delta > 0$ since $\alpha_{\varepsilon,\delta}$ converges to $\alpha_{0,0}$. Since Δ_K is bounded, we can obtain $\gamma_{\varepsilon,\delta} \in C^{q-1}(\mathcal{U}, \omega_X \otimes F \otimes \mathcal{J}(h))$ such that

$$\mu\gamma_{\varepsilon,\delta} = \alpha_{\varepsilon,\delta} \quad \text{and} \quad p_{K_{i_0\dots i_{q-1}}}(\gamma_{\varepsilon,\delta})^2 \leq C_K$$

for some positive constant C_K . The above constant C_K depends on the choice of K , γ , but does not depend on ε, δ .

By the same argument as in [MaS4, Claim 5.11 and Claim 5.13], we can obtain F -valued $(n, q-1)$ -forms $w_{\varepsilon,\delta}$ with the desired properties. The strategy is as follows: The inverse map $\overline{g_{\varepsilon,\delta}}$ of $\overline{f_{\varepsilon,\delta}}$ is explicitly constructed by using a partition of unity (see the proof of [MaS4, Proposition 5.5] and [MaS4, Remark 5.6]). We can easily see that $g_{\varepsilon,\delta}(\mu\gamma_{\varepsilon,\delta}) = \bar{\partial}v_{\varepsilon,\delta}$ and $g_{\varepsilon,\delta}(\alpha_{\varepsilon,\delta}) = U_{\varepsilon,\delta} + \bar{\partial}\tilde{v}_{\varepsilon,\delta}$ hold for some $v_{\varepsilon,\delta}$ and $\tilde{v}_{\varepsilon,\delta}$ by the De Rham–Weil isomorphism. In particular, we have $U_{\varepsilon,\delta} = \bar{\partial}(v_{\varepsilon,\delta} - \tilde{v}_{\varepsilon,\delta})$ by $\mu\gamma_{\varepsilon,\delta} = \alpha_{\varepsilon,\delta}$. The important point here is that we can explicitly compute $v_{\varepsilon,\delta}$ and $\tilde{v}_{\varepsilon,\delta}$ by using the partition of unity, $\beta_{i_0\dots i_k}^{\varepsilon,\delta}$, and $\gamma_{\varepsilon,\delta}$. From this explicit expression, we obtain the L^2 -estimate for $v_{\varepsilon,\delta}$ and $\tilde{v}_{\varepsilon,\delta}$. See [MaS4, Claim 5.11 and 5.13] for the precise argument. \square

Proposition 5.10. *There exist $F \otimes M$ -valued $(n, q-1)$ -forms $v_{\varepsilon, \delta}$ on Y_ε with the following properties:*

- $\overline{\partial}v_{\varepsilon, \delta} = su_{\varepsilon, \delta}$.
- $\lim_{\delta \rightarrow 0} \|v_{\varepsilon, \delta}\|_{\varepsilon, \delta}$ can be bounded by a constant independent of ε .

Proof of Proposition 5.10. Since the cohomology class of su is assumed to be zero in $H^q(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes M)$, there exists an $F \otimes M$ -valued $(n, q-1)$ -form v such that $\overline{\partial}v = su$ and $\|v\|_{h, \omega} < \infty$. For $w_{\varepsilon, \delta}$ satisfying the properties in Proposition 5.9, by putting $v_{\varepsilon, \delta} := -sw_{\varepsilon, \delta} + v$, we have $\overline{\partial}v_{\varepsilon, \delta} = su_{\varepsilon, \delta}$. Furthermore, an easy computation yields

$$\|v_{\varepsilon, \delta}\|_{\varepsilon, \delta} \leq \|sw_{\varepsilon, \delta}\|_{\varepsilon, \delta} + \|v\|_{\varepsilon, \delta} \leq \sup_X |s|_{h_M} \|w_{\varepsilon, \delta}\|_{\varepsilon, \delta} + \|v\|_{\varepsilon, \delta}.$$

By Lemma 2.4, property (b), and property (B), we have $\|v\|_{\varepsilon, \delta} \leq \|v\|_{h, \omega} < \infty$. This completes the proof. \square

The proof of Theorem 5.1 is completed by the following proposition (see Proposition 5.7).

Proposition 5.11.

$$\lim_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \|su_{\varepsilon, \delta}\|_{\varepsilon, \delta} = 0.$$

Proof of Proposition 5.11. For the solution $v_{\varepsilon, \delta}$ satisfying the properties in Proposition 5.10, it is easy to see

$$\lim_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \|su_{\varepsilon, \delta}\|_{\varepsilon, \delta}^2 = \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \langle \overline{\partial}_{\varepsilon, \delta}^* su_{\varepsilon, \delta}, v_{\varepsilon, \delta} \rangle_{\varepsilon, \delta} \leq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{\delta \rightarrow 0} \|\overline{\partial}_{\varepsilon, \delta}^* su_{\varepsilon, \delta}\|_{\varepsilon, \delta} \|v_{\varepsilon, \delta}\|_{\varepsilon, \delta}.$$

Proposition 5.8 and Proposition 5.10 assert that the right hand side is zero. \square

We close this step with the following lemma:

Lemma 5.12 (cf. [D1, 4.1 Théorème]). *Assume that B is a Stein open set in X such that $\omega_{\varepsilon, \delta} = \sqrt{-1}\partial\overline{\partial}(\Psi + \delta\Psi_\varepsilon)$ on a neighborhood of \overline{B} . Then, for an arbitrary $\alpha \in \text{Ker } \overline{\partial} \subset L_{(2)}^{n, q}(B \setminus Z_\varepsilon, F)_{\varepsilon, \delta}$, there exist $\beta \in L_{(2)}^{n, q-1}(B \setminus Z_\varepsilon, F)_{\varepsilon, \delta}$ and a positive constant $C_{\varepsilon, \delta}$ (independent of α) such that*

- $\overline{\partial}\beta = \alpha$ and $\|\beta\|_{\varepsilon, \delta}^2 \leq C_{\varepsilon, \delta} \|\alpha\|_{\varepsilon, \delta}^2$,
- $\overline{\lim}_{\delta \rightarrow 0} C_{\varepsilon, \delta}$ (is finite and) is independent of ε .

Proof of Lemma 5.12. We may assume $\varepsilon < 1/2$. For the singular hermitian metric $H_{\varepsilon, \delta}$ on F defined by $H_{\varepsilon, \delta} := h_\varepsilon e^{-(\Psi + \delta\Psi_\varepsilon)}$, the curvature satisfies

$$\sqrt{-1}\Theta_{H_{\varepsilon, \delta}}(F) = \sqrt{-1}\Theta_{h_\varepsilon}(F) + \sqrt{-1}\partial\overline{\partial}(\Psi + \delta\Psi_\varepsilon) \geq -\varepsilon\omega + \omega_{\varepsilon, \delta} \geq (1 - \varepsilon)\omega_{\varepsilon, \delta} \geq \frac{1}{2}\omega_{\varepsilon, \delta}$$

by property (B) and $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon\omega$. The L^2 -norm $\|\alpha\|_{H_{\varepsilon, \delta}, \omega_{\varepsilon, \delta}}$ with respect to $H_{\varepsilon, \delta}$ and $\omega_{\varepsilon, \delta}$ is finite since the function $\Psi + \delta\Psi_\varepsilon$ is bounded and $\|\alpha\|_{\varepsilon, \delta}$ is finite. Therefore, from the standard L^2 -method for the $\overline{\partial}$ -equation (for example see [D1, 4.1 Théorème]), we obtain a solution β of the $\overline{\partial}$ -equation $\overline{\partial}\beta = \alpha$ with

$$\|\beta\|_{H_{\varepsilon, \delta}, \omega_{\varepsilon, \delta}}^2 \leq \frac{2}{q} \|\alpha\|_{H_{\varepsilon, \delta}, \omega_{\varepsilon, \delta}}^2.$$

Then we can easily see

$$\|\beta\|_{\varepsilon, \delta}^2 \leq \frac{2 \sup_B e^{-(\Psi + \delta \Psi_\varepsilon)}}{q \inf_B e^{-(\Psi + \delta \Psi_\varepsilon)}} \|\alpha\|_{\varepsilon, \delta}^2.$$

This completes the proof. \square

Remark 5.13. In Lemma 5.12, we take a solution $\beta_0 \in L_{(2)}^{n, q-1}(B \setminus Z_\varepsilon, F)_{\varepsilon, \delta}$ of the equation $\bar{\partial}\beta = \alpha$. Then β_0 is uniquely decomposed as follows:

$$\beta_0 = \beta_1 + \beta_2 \quad \text{for } \beta_1 \in \text{Ker } \bar{\partial} \text{ and } \beta_2 \in (\text{Ker } \bar{\partial})^\perp.$$

We can easily check that β_2 is a unique solution of $\bar{\partial}\beta = \alpha$ whose norm is minimum among all the solutions.

Thus we finish the proof of Theorem 5.1. \square

6. TWISTS BY NAKANO SEMIPOSITIVE VECTOR BUNDLES

We have already known that some results for ω_X can be generalized for $\omega_X \otimes E$, where E is a Nakano semipositive vector bundle on X (see, for example, [Ta], [Mo], and [Fs]). Let us recall the definition of Nakano semipositive vector bundles.

Definition 6.1 (Nakano semipositive vector bundles). Let E be a holomorphic vector bundle on a complex manifold X . If E admits a smooth hermitian metric h_E such that the curvature form $\sqrt{-1}\Theta_{h_E}(E)$ defines a positive semi-definite hermitian form on each fiber of the vector bundle $E \otimes T_X$, where T_X is the holomorphic tangent bundle of X , then E is called a Nakano semipositive vector bundle.

Example 6.2 (Unitary flat vector bundles). Let E be a holomorphic vector bundle on a complex manifold X . If E admits a smooth hermitian metric h_E such that (E, h_E) is flat, that is, $\sqrt{-1}\Theta_{h_E}(E) = 0$, then E is Nakano semipositive.

We need the following two easy lemmas on Nakano semipositive vector bundles.

Lemma 6.3. *Let E be a Nakano semipositive vector bundle on a complex manifold X . Let H be a smooth divisor on X . Then $E|_H$ is a Nakano semipositive vector bundle on H .*

Proof. This lemma easily follows from the definition of Nakano semipositive vector bundles (see Definition 6.1). \square

Lemma 6.4. *Let $q : Z \rightarrow X$ be an étale morphism between complex manifolds. Let (E, h_E) be a Nakano semipositive vector bundle on X . Then (q^*E, q^*h_E) is a Nakano semipositive vector bundle on Z .*

Proof. It is obvious by the definition of Nakano semipositive vector bundles (see Definition 6.1). \square

Proposition 6.5. *Proposition 1.9 holds even when ω_X is replaced with $\omega_X \otimes E$, where E is a Nakano semipositive vector bundle on X .*

Proof. By Lemma 6.3 and Lemma 6.4, the proof of Proposition 1.9 in Section 4 works for $\omega_X \otimes E$. \square

Therefore, by Proposition 6.5 and the proof of Theorem 1.4 and Corollary 1.7 in Section 4, it is sufficient to prove the following theorem for Theorem 1.11.

Theorem 6.6 (Theorem A twisted by Nakano semipositive vector bundles). *Let E be a Nakano semipositive vector bundle on a compact Kähler manifold X . Let F (resp. M) be a line bundle on a compact Kähler manifold X with a singular hermitian metric h (resp. a smooth hermitian metric h_M) satisfying*

$$\sqrt{-1}\Theta_{h_M}(M) \geq 0 \text{ and } \sqrt{-1}\Theta_h(F) - b\sqrt{-1}\Theta_{h_M}(M) \geq 0 \text{ for some } b > 0.$$

Then, for a (nonzero) section $s \in H^0(X, M)$, the multiplication map induced by $\otimes s$

$$H^q(X, \omega_X \otimes E \otimes F \otimes \mathcal{J}(h)) \xrightarrow{\otimes s} H^q(X, \omega_X \otimes E \otimes F \otimes \mathcal{J}(h) \otimes M)$$

is injective for every q . Here ω_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h .

We will explain how to modify the proof of Theorem 5.1 for Theorem 6.6.

Proof. We replace (F, h_ε) with $(E \otimes F, h_E h_\varepsilon)$ in the proof of Theorem 5.1, where $\{h_\varepsilon\}_{1 \gg \varepsilon > 0}$ is a family of singular hermitian metrics on F (constructed in Step 1) and h_E is a smooth hermitian metric on E such that $\sqrt{-1}\Theta_{h_E}(E)$ is Nakano semipositive. Then it is easy to see that essentially the same proof as in Theorem 5.1 works for Theorem 6.6 thanks to the assumption on the curvature of E . For the reader's convenience, we give several remarks on the differences with the proof of Theorem 5.1.

There is no problem when we construct h_ε and $\omega_{\varepsilon, \delta}$. In Step 4 in the proof of Theorem 5.1, we used the de Rham–Weil isomorphism (see (5.7) and [MaS4, Proposition 5.5]), which was constructed by using Lemma 5.12. Since [D1, 4.1 Théorème] (which yields Lemma 5.12) is formulated for holomorphic vector bundles, Lemma 5.12 can be generalized to $(E \otimes F, h_E h_\varepsilon)$. From this generalization, we can construct the de Rham–Weil isomorphism for $E \otimes F$

$$\overline{f_{\varepsilon, \delta}}: \frac{\text{Ker } \overline{\partial}}{\text{Im } \overline{\partial}} \text{ of } L_{(2)}^{n, q}(E \otimes F)_{h_E h_\varepsilon, \omega_{\varepsilon, \delta}} \xrightarrow{\cong} \frac{\text{Ker } \mu}{\text{Im } \mu} \text{ of } C^q(\mathcal{U}, \omega_X \otimes E \otimes F \otimes \mathcal{J}(h_\varepsilon)).$$

In Step 1, we used the orthogonal decomposition of $L_{(2)}^{n, q}(F)_{\varepsilon, \delta}$, which was obtained from the fact that $\text{Im } \overline{\partial} \subset L_{(2)}^{n, q}(F)_{\varepsilon, \delta}$ is closed. To obtain the same conclusion for $L_{(2)}^{n, q}(E \otimes F)_{h_E h_\varepsilon, \omega_{\varepsilon, \delta}}$, it is sufficient to show that $C^q(\mathcal{U}, K_X \otimes E \otimes F \otimes \mathcal{J}(h_\varepsilon))$ is a Fréchet space (see [MaS4, Proposition 5.8]). We can easily check it by using the same argument as in [MaS4, Theorem 5.3] for $\mathbb{C}^{\text{rank } E}$ -valued holomorphic functions.

The argument of Step 2 works even if we consider $(E \otimes F, h_E h_\varepsilon)$. In Step 3, we need to prove (5.6), but it is easy to see

$$\begin{aligned} -\varepsilon q |u_{\varepsilon, \delta}|_{h_E h_\varepsilon, \omega_{\varepsilon, \delta}}^2 &\leq \langle \sqrt{-1}\Theta_{h_\varepsilon}(F) \Lambda_{\omega_{\varepsilon, \delta}} u_{\varepsilon, \delta}, u_{\varepsilon, \delta} \rangle_{h_E h_\varepsilon, \omega_{\varepsilon, \delta}} \\ &\leq \langle \sqrt{-1}\Theta_{h_E h_\varepsilon}(E \otimes F) \Lambda_{\omega_{\varepsilon, \delta}} u_{\varepsilon, \delta}, u_{\varepsilon, \delta} \rangle_{h_E h_\varepsilon, \omega_{\varepsilon, \delta}} \end{aligned}$$

since $\sqrt{-1}\Theta_{h_E}(E)$ is Nakano semipositive. □

When E is Nakano semipositive and is not flat, there seems to be no Hodge theoretic approach to Theorem 6.6 even if h is smooth. We note that Theorem 6.6 follows from [F4, Theorem 1.2], which is analytic, when h is smooth on a nonempty Zariski open set.

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